**Math 375, Spring 2018**
**Professor Levandosky**
**Midterm Exam 3 Solutions**

1. Suppose $Y$ is a random variable with probability density function

$$f(y) = \begin{cases} cy^{-5} & y \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find $c$.
   **Solution.** Since
$$\int_{-\infty}^{\infty} f(y)\,dy = \int_{1}^{\infty} cy^{-5}\,dy = \frac{1}{4}c,$$
   $c$ must be 4.

   (b) Find the distribution function $F(y)$ for $Y$.
   **Solution.** $F(y) = \int_{-\infty}^{y} f(t)\,dt$, so for $y < 1$, $F(y) = \int_{-\infty}^{y} 0\,dt = 0$, and for $y \geq 1$, $F(y) = \int_{1}^{y} 4t^{-5}\,dt = 1 - y^{-4}$. So
$$F(y) = \begin{cases} 0 & y < 1 \\ 1 - y^{-4} & y \geq 1 \end{cases}$$

   (c) Find $P(Y \leq 3)$.
   **Solution.** This is just $F(3) = 1 - (3)^{-4} = \frac{80}{81}$.

   (d) Find the median of $Y$.
   **Solution.** The median $\phi_{1/2}$ satisfies $F(\phi_{1/2}) = \frac{1}{2}$, so solving $1 - (\phi_{1/2})^{-4} = \frac{1}{2}$ gives $\phi_{1/2} = 2^{1/4}$.

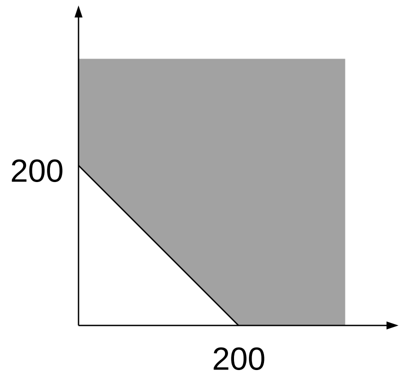
   (e) Find $E(Y)$.
   **Solution.** $E(Y) = \int_{-\infty}^{\infty} yf(y)\,dy = \int_{1}^{\infty} 4y^{-4}\,dy = \frac{4}{3}$.

   (f) Find $V(Y)$.
   **Solution.** $E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_{1}^{\infty} 4y^{-3}\,dy = 2$, so $V(Y) = E(Y^2) - E(Y)^2 = \frac{2}{9}$.

2. Suppose the life span of a type of electronic component has exponential distribution with mean 100 hours.

   (a) Find the probability such a component lasts at least 100 hours.
   **Solution.** The probability density for the life span $Y$ is $f(y) = \frac{1}{100}e^{-y/100}$ for $y \geq 0$. Thus
$$P(Y \geq 100) = \int_{100}^{\infty} \frac{1}{100}e^{-y/100}\,dy = e^{-1}.$$

   (b) Suppose two identical components have independent life spans $Y_1$ and $Y_2$ with exponential distributions with mean 100 hours, and one is used as a backup for the other. Find the probability that their total life span $Y_1 + Y_2$ exceeds 200 hours.
   **Solution.** The joint density for $Y_1$ and $Y_2$ is
$$f(y_1, y_2) = \begin{cases} \frac{1}{10000}e^{-y_1/100}e^{-y_2/100} & y_1 \geq 0, y_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
   To find $P(Y_1 + Y_2 > 200)$ directly, one could integrate $f$ over the region in the first quadrant above the line $y_1 + y_2 = 200$.




Since this must be split up into two integrals

$$\int_0^{200} \int_{200-y_1}^{\infty} f(y_1, y_2)\, dy_2\, dy_1 + \int_{200}^{\infty} \int_0^{\infty} f(y_1, y_2)\, dy_2\, dy_1$$

it is simpler to first compute $P(Y_1+Y_2 \leq 200)$, which is the integral of $f$ over the triangle below the line $y_1 + y_2 = 200$ in the first quadrant.

$$\begin{aligned}
P(Y_1 + Y_2 \leq 100) &= \int_0^{200} \int_0^{200-y_1} f(y_1, y_2)\, dy_2\, dy_1 \\
&= \int_0^{200} -\frac{1}{100} e^{-y_1/100} e^{-y_2/100} \Big|_0^{200-y_1} dy_1 \\
&= \int_0^{200} -\frac{1}{100} e^{-y_1/100} \left(e^{-2+y_1/100} - 1\right) dy_1 \\
&= \int_0^{200} \frac{1}{100} \left(e^{-y_1/100} - e^{-2}\right) dy_1 \\
&= 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}.
\end{aligned}$$

Thus $P(Y_1 + Y_2 > 200) = 3e^{-2}$.

3. According to marathonguide.com, the average marathon time among all marathons in the United States was 4 hours and 38 minutes, with a standard deviation of 1 hour and 2 minutes. For this question, assume marathon times are normally distributed.

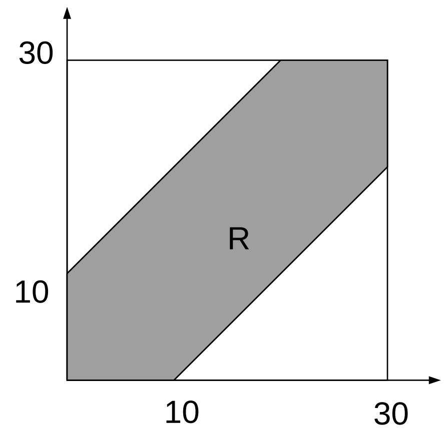
   (a) Find the probability that a randomly selected marathon finisher ran under four hours.
   **Solution.** Let $Y$ be the finishing time in minutes. Then $Z = \frac{Y-278}{62}$ is standard normal, so

   $$P(Y < 240) = P(Z < -\frac{38}{62} = P(Z < -0.61) = P(Z > 0.61) = 0.2709$$

   (b) What time (to the nearest minute) would someone need to run to be in the fastest 5% of marathon times?
   **Solution.** $P(Z \leq z_0) = 0.5$ when $z_0 = -1.65$, and solving $\frac{Y-278}{62} = -1.65$ gives $Y = 175.7$, or about 2 hours and 56 minutes.

4. Two friends agree to meet at the park for a run between 6:00 and 6:30. Their arrival times are independent and uniform during this time period. Each will wait up to 10 minutes for the other. What is the probability that they will meet?

**Solution.** Let $Y_1$ and $Y_2$ denote the number of minutes after 6:00 that each person arrives. Then the joint density for $Y_1$ and $Y_2$ is $f(y_1, y_2) = \frac{1}{900}$ on the square $[0, 30] \times [0, 30]$. They will meet if $Y_1 - 10 \leq Y_2 \leq Y_1 + 10$. This event corresponds to the shaded region in the figure below.




Thus the probability that they meet is

$$P(Y_1 - 10 \leq Y_2 \leq Y_1 + 10) = \iint_R \frac{1}{900}\, dA = \frac{1}{900}\,\text{area}(R) = \frac{1}{900}(500) = \frac{5}{9}.$$

5. Suppose $Y_1$ and $Y_2$ have joint density function defined by $f(y_1, y_2) = y_1 + y_2$ on the domain $D = \{(y_1, y_2) : 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1\}$, and $f(y_1, y_2) = 0$ elsewhere.

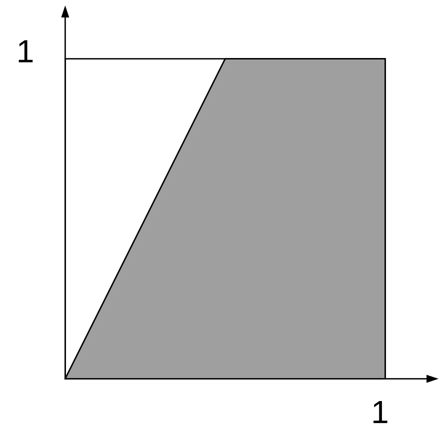
   (a) Find $P(Y_1 \leq \frac{1}{2})$.
   **Solution.**

$$P\left(Y_1 \leq \frac{1}{2}\right)) = \int_0^1 \int_0^{1/2} y_1 + y_2\, dy_1\, dy_2 = \int_0^1 \frac{1}{8} + \frac{1}{2} y_2\, dy_2 = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

   (b) Find the marginal densities $f_1(y_1)$ and $f_2(y_2)$.
   **Solution.**

$$f_1(y_1) = \int_0^1 y_1 + y_2\, dy_2 = y_1 + \frac{1}{2}$$
$$f_2(y_2) = \int_0^1 y_1 + y_2\, dy_1 = \frac{1}{2} + y_2$$

   (c) Find $P(Y_2 \leq 2Y_1)$.
   **Solution.** The event $Y_2 \leq 2Y_1$ corresponds to the region inside the square $[0, 1] \times [0, 1]$ below the line $y_2 = 2y_1$.




It is simpler to find $P(Y_2 \geq 2Y_1)$ since the event $Y_2 \geq 2Y_1$ corresponds to the triangular region above the line $y_2 = 2y_1$. This triangle is $\{(y_1, y_2) : 0 \leq y_2 \leq 1, 0 \leq y_1 \leq \frac{1}{2}y_2\}$. Thus

$$P(Y_2 \geq 2Y_1) = \int_0^1 \int_0^{y_2/2} y_1 + y_2 \, dy_1 \, dy_2 = \int_0^1 \frac{1}{8}y_2^2 + \frac{1}{2}y_2^2 \, dy_2 = \int_0^1 \frac{5}{8}y_2^2 \, dy_2 = \frac{5}{24}$$

and $P(Y_2 \leq 2Y_1) = \frac{19}{24}$.

(d) Find $P(Y_1 \leq \frac{1}{2}|Y_2 = 1)$.

**Solution.** The conditional density of $Y_1$ given $Y_2$ is $f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)} = \frac{y_1+y_2}{\frac{1}{2}+y_2}$. Thus

$$P\left(Y_1 \leq \frac{1}{2}|Y_2 = 1\right) = \int_0^1 f(y_1|1) \, dy_1 = \int_0^{1/2} \frac{y_1 + 1}{\frac{3}{2}} \, dy_1 = \frac{5/8}{3/2} = \frac{10}{24}.$$

(e) Find $E(Y_1)$ and $E(Y_2)$.

**Solution.**

$$E(Y_1) = \int_0^1 \int_0^1 y_1(y_1+y_2) \, dy_2 \, dy_1 = \int_0^1 \int_0^1 y_1^2+y_1y_2 \, dy_2 \, dy_1 = \int_0^1 y_1^2+\frac{1}{2}y_1 \, dy_1 = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$$

Likewise $E(Y_2) = \frac{7}{12}$.

(f) Are $Y_1$ and $Y_2$ independent? Explain.

**Solution.** By part (b) $f_1(y_1)f_2(y_2) = (\frac{1}{2}+y_1)(\frac{1}{2}+y_2)$. Since this is not equal to $f(y_1, y_2)$, $Y_1$ and $Y_2$ are not independent.

6. Suppose $E(Y_1) = 5$, $V(Y_1) = 2$, $E(Y_2) = 6$, $V(Y_2) = 1$, and $E(Y_1Y_2) = 3$. Let $Z = 3Y_1 - Y_2$.

(a) Find $\text{Cov}(Y_1, Y_2)$.

**Solution.** $\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 3 - 30 = -27$

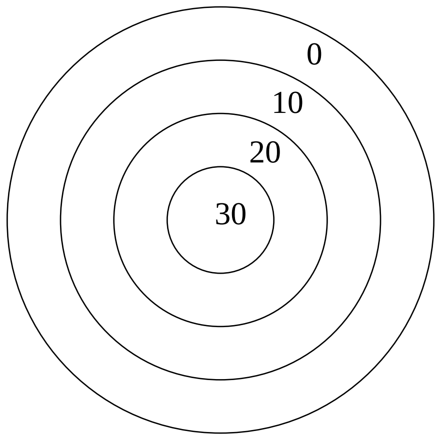
(b) Find $E(Z)$.

**Solution.** $E(Z) = 3E(Y_1) - E(Y_2) = 9$.

(c) Find $V(Z)$.

**Solution.** $V(Z) = V(3Y_1 - Y_2) = 9V(Y_1) - 6\,\text{Cov}(Y_1, Y_2) + V(Y_2) = 9(2) - 6(-27) + 1 = 181$

7. Suppose an archery target has radius 40 centimeters. Within it are concentric circles with radii 10cm, 20cm and 30cm. An arrow scores 30 points if it hits the inner most circle, 20 points if hits the next ring, 10 points for the next ring, and 0 points for the outer ring. Suppose each shot hits a point uniformly distributed on the target, and that the shots are independent of one another.




(a) Find the expected value of the score after one shot.

**Solution.** The possible score after one shot is a random variable $Y$ with values 0, 10, 20, and 30. The area of the target is $1600\pi$, so the uniform density over this target is $f(y_1, y_2) = \frac{1}{1600\pi}$. The inner circle $C$ has area $100\pi$. Thus the probability of hitting it is

$$\iint_C \frac{1}{1600\pi}\, dy_1\, dy_2 = \frac{1}{1600\pi}\,\text{area}(C) = \frac{100\pi}{1600\pi} = \frac{1}{16}$$

and thus $p(30) = \frac{1}{16}$. The 20 point ring $R$ has area $400\pi - 100\pi = 300\pi$, so the probability of hitting it is

$$\iint_R \frac{1}{1600\pi}\, dy_1\, dy_2 = \frac{1}{1600\pi}\,\text{area}(R) = \frac{300\pi}{1600\pi} = \frac{3}{16}$$

and thus $p(20) = \frac{3}{16}$. Similarly, $p(10) = \frac{5}{16}$ and $p(0) = \frac{7}{16}$. Thus the expected value of $Y$ is

$$E(Y) = 0p(0) + 10p(10) + 20p(20) + 30p(30) = \frac{140}{16} = 8.75$$

(b) Find the probability that the score after two shots is 20.

**Solution.** The score can be 20 after two shots in three ways: (0,20), (20,0) or (10,10) Since the shots are independent, $p(0, 20) = p(20, 0) = p(0)p(20) = \frac{21}{256}$ and $p(10, 10) = p(10)^2 = \frac{25}{256}$. Thus the probability of a total score of 20 after two shots is $2\cdot\frac{21}{256} + \frac{25}{256} = \frac{67}{256}$.

8. Recall that the moment generating function for a standard normal random variable $Z$ is $m(t) = e^{t^2/2}$.

(a) Find $E(Z^3)$.

**Solution.** We know $E(Y^3) = m'''(0)$. One could calculate this by computing $m'''(t)$ and evaluating at $t = 0$. Instead, notice that the Taylor series for $e^x$ at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

This implies that the Taylor series for $m(t)$ at $t = 0$ is

$$m(t) = e^{t^2/2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!} = 1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 + \frac{1}{48}t^6 + \cdots$$

By definition of a Taylor series, the coefficient of $t^3$ in this series must be $m'''(0)/3!$. But there is no $t^3$ term, so this coefficient is zero, hence $E(Y^3) = m'''(0) = 0$.

(b) Find $E(Z^4)$.

**Solution.** In the series for $m(t)$ above, the coefficient of $t^4$ must be $m''''(0)/4!$. This coefficient is $\frac{1}{8}$, and thus $E(Y^4) = m''''(0) = \frac{4!}{8} = 3$.

(c) Bonus: Find a general formula for $E(Z^n)$.

**Solution.** Since the series for $m(t)$ only contains even powers of $t$, all of the odd moments will be zero. The coefficient of $t^{2n}$ is $\frac{1}{2^n n!}$, but this coefficient is also $\frac{f^{(2n)}(0)}{(2n)!}$, so $E(Y^{2n}) = f^{(2n)}(0) = \frac{(2n)!}{2^n n!}$.