Math 375, Spring 2018
Professor Levandosky
Midterm Exam 2 Solutions

1. For each random variable, write a formula for its probability distribution function $p(y)$, and find its mean μ and standard deviation σ.

   (a) Binomial with $n = 12$ and $p = 0.3$.

   Solution. $p(y) = \binom{12}{y}(0.3)^y(0.7)^{12-y}$, $E(Y) = 12(0.3) = 3.6$ and $\sigma = \sqrt{12(0.3)(0.7)} \approx 1.587$

   (b) Negative binomial with $p = 4/5$ and $r = 4$.

   Solution. $p(y) = \binom{y-1}{3}(4/5)^4(1/5)^{y-4}$, $E(Y) = r/p = 5$ and $V(Y) = r(1-p)/p^2 = \frac{5}{4}$

   (c) Poisson with $\lambda = 6$.

   Solution. $p(y) = \frac{6^y e^{-6}}{y!}$, $E(Y) = 6$ and $\sigma = \sqrt{6}$.

   (d) Hypergeometric with $N = 10$, $n = 7$ and $r = 6$.

   Solution. $p(y) = \dfrac{\binom{6}{y}\binom{4}{7-y}}{\binom{10}{7}}$, $E(Y) = \frac{nr}{N} = 4.2$ and $\sigma = \sqrt{7 \cdot \frac{6}{10} \cdot \frac{4}{10} \cdot \frac{3}{9}} = \sqrt{0.56} \approx 0.748$

2. Suppose Lisa is an 80% free-throw shooter.

   (a) First suppose Lisa takes 5 free-throws.

   (i) What is the probability she makes at least 3 of them?
   Solution. Let Y denote the number of free-throws Lisa makes. Then Y is binomial with $n = 5$ and $p = 0.8$, so the probability she makes at least 3 free-throws is

$$
\begin{aligned}
P(Y \geq 3) &= p(3) + p(4) + p(5) \\
&= \binom{5}{3}(0.8)^3(0.2)^2 + \binom{5}{4}(0.8)^4(0.2) + \binom{5}{5}(0.8)^5 \\
&= 0.2048 + 0.4096 + 0.32768 \\
&= 0.94208
\end{aligned}
$$

   (ii) What is the probability she makes all 5 given that she makes at least 3?
   Solution.

$$
\begin{aligned}
P(Y = 5 | Y \geq 3) &= \frac{P((Y = 5) \cap (Y \geq 3))}{P(Y \geq 3)} \\
&= \frac{P(Y = 5)}{P(Y \geq 3)} \\
&= \frac{0.32768}{0.94208} \\
&\approx 0.3478
\end{aligned}
$$

(b) Now suppose Lisa takes free-throws until she makes 7.

  (i) What is the expected number of shots she must take?
  Solution. Let Y denote the number of shots she must take to make 7. Then Y is negative binomial with $r = 7$ and $p = 0.8$, so $E(Y) = r/p = 8.75$.

  (ii) What is the probability that it takes her exactly 10 shots to do this?
  Solution. $P(Y = 10) = \binom{9}{6}(0.8)^7(0.2)^3 \approx 0.141$

3. In the game Scrabble there are 100 tiles, consisting of 54 consonants, 44 vowels, and two blank tiles. The game begins with each person drawing 7 tiles (without replacement). Suppose Katie draws first.

(a) What is the probability that she will draw 5 or more consonants? (You don't need to simplify your answer.)

Solution. Let Y denote the number of consonants Katie draws. Then Y is hypergeometric with $N = 100$, $n = 7$ and $r = 54$, so

$$P(Y \geq 5) = p(5) + p(6) + p(7) = \frac{\binom{54}{5}\binom{46}{2} + \binom{54}{6}\binom{46}{1} + \binom{54}{7}\binom{46}{0}}{\binom{100}{7}}$$

(b) If it is known that Katie drew at least 5 consonants what is the probability that all 7 of her tiles are consonants?

Solution.

$$P(Y = 7|Y \geq 5) = \frac{P(Y = 7)}{P(Y \geq 5)} = \frac{\binom{54}{7}\binom{46}{0}}{\binom{54}{5}\binom{46}{2} + \binom{54}{6}\binom{46}{1} + \binom{54}{7}\binom{46}{0}}$$

(c) What is the expected number of consonants Katie draws?

Solution. $E(Y) = \frac{nr}{N} = \frac{(7)(54)}{100} = 3.78$.

4. Suppose Sarah receives an average of 2.6 texts per hour.

(a) Let Y be the number of texts Sarah receives in a given hour. What distribution best models Y? Write a formula for the distribution function $p(y)$.

Solution. Poisson with $\lambda = 2.6$, $p(y) = \frac{(2.6)^y e^{-2.6}}{y!}$.

(b) What is the probability that Sarah receives **exactly 3** texts between 2PM and 3PM?

Solution. $p(3) = \frac{(2.6)^3 e^{-2.6}}{3!} \approx 0.2175$.

(c) What is the probability there she receives **at least 5** texts between 3PM and 5PM?

Solution. Let Y denote the number of texts received in a 2 hour window. Then Y is Poisson with $\lambda = 5.2$. Using Table 3, $P(Y \leq 5) = 0.581$, so $P(Y \geq 5) = 0.419$.

5. Suppose a random variable Y has moment generating function $m(t) = (\frac{1}{3}e^t + \frac{2}{3})^7$.

(a) Find $P(Y = 3)$.
Solution. This is the moment generating function for a binomial random variable with $n = 7$ and $p = \frac{1}{3}$, so $P(Y = 3) = \binom{7}{3}(1/3)^3(2/3)^4 \approx 0.256$.

(b) Find $E(Y)$ and $E(Y^2)$.
Solution. $E(Y) = \frac{7}{3}$ and $E(Y^2) = V(Y) + E(Y)^2 = \frac{14}{9} + \left(\frac{7}{3}\right)^2 = 7$.

6. Suppose a random variable Y has mean 50 and variance 16.

(a) Find a lower bound on $P(40 < Y < 60)$.
Solution. Since $\sigma = 4$ and $10 = 2.5\sigma$, Chebyshev's inequality implies

$$P(40 < Y < 60) = P(|Y - 50| < 10) = P(|Y - \mu| < 2.5\sigma) \geq 1 - \frac{1}{(2.5)^2} = 0.84$$

(b) For what C is $P(|Y - 50| < C) \geq 0.99$?
Solution. Since $0.99 = 1 - \frac{1}{10^2}$, Checbyshev's inequality implies $P(|Y - 50| < 10\sigma) \geq 0.99$, so $C = 10\sigma = 40$.

7. Suppose Y has geometric distribution with probability of success p, so $p(y) = pq^{y-1}$, where $q = 1 - p$.

(a) Show that the probability generating function of Y is $P(t) = \dfrac{pt}{1 - qt}$.

Solution.

$$P(t) = E(t^y) = \sum_{y=1}^{\infty} t^y pq^{y-1} = \sum_{y=1}^{\infty} pq^{-1}(tq)^y.$$

This is a geometric series with ratio $r = tq$ and first term $a = pt$, so (provided $|t| < 1/q$) its sum is

$$\frac{a}{1 - r} = \frac{pt}{1 - qt}$$

and thus $P(t) = \frac{pt}{1-qt}$.

(b) Show that the moment generating function of Y is $m(t) = \dfrac{pe^t}{1 - qe^t}$.

Solution. Using part (a), $m(t) = P(e^t) = \frac{pe^t}{1-qe^t}$.

(c) Show that $E(Y(Y - 1)(Y - 2)) = \dfrac{6q^2}{p^3}$, where $q = 1 - p$.

Solution. Since $P'(t) = p(1 - qt)^{-2}$, $P''(t) = 2qp(1 - qt)^{-3}$, and $P'''(t) = 6q^2p(1 - qt)^{-4}$, it follows that

$$E(Y(Y - 1)(Y - 2)) = P'''(1) = 6q^2p(1 - q)^{-4} = 6q^2p(p)^{-4} = 6q^2p^{-3}.$$

(d) Find $E(Y^3)$.
Solution. By part (c) we know $\frac{6q^2}{p^3} = E(Y(Y - 1)(Y - 2)) = E(Y^3) - 3E(Y^2) + 2E(Y)$, so

$$E(Y^3) = \frac{6q^2}{p^3} + 3E(Y^2) - 2E(Y) = \frac{6q^2}{p^3} + 3E(Y(Y - 1)) + E(Y)$$

Now $E(Y) = P'(1) = \frac{1}{p}$ and $E(Y(Y-1)) = P''(1) = \frac{2q}{p^2}$, so

$$\begin{aligned} E(Y^3) &= \frac{6q^2}{p^3} + 3 \cdot \frac{2q}{p^2} + \frac{1}{p} \\ &= \frac{6q^2 + 6qp + p^2}{p^3} \\ &= \frac{p^2 - 6p + 6}{p^3} \end{aligned}$$