College of the Holy Cross, Fall 2018
Math 244, Midterm 3

1. Let $A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 0 & 3 \\ 7 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$.

   (a) Which of the following matrix products are defined? Compute those that are.

   (i) AB
   (ii) AC
   (iii) BA
   (iv) BC
   (v) CA
   (vi) CB

   (b) Let

$$S(x_1, x_2, x_3) = (2x_1 + x_2 + 4x_3, -2x_1 + 4x_2 + x_3)$$
$$T(y_1, y_2) = (4y_1 + 2y_2, 3y_2, 7y_1 - y_2).$$

   Which of the matrices in part (a) is the matrix for $T \circ S$ with respect to the standard basis for $\mathbf{R}^3$?

2. Let $S : U \to V$ and $T : V \to W$ be linear transformations.

   (a) Prove that if S and T are surjective, then $T \circ S$ is surjective.

   (b) Prove that if $T \circ S$ is surjective, then T is surjective.

3. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 4 & 0 \\ -2 & 1 & 1 \end{bmatrix}$.

   (a) Find A^{-1}.

   (b) Find the solution of $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

   (c) Suppose $T : \mathbf{R}^3 \to \mathbf{R}^3$ is the linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

   Let B be the matrix for T with respect to the standard basis. Find B.

4. (a) Complete the following definition. The matrices A and B are **similar** if

   (b) Suppose A and B are invertible matrices that are similar. Prove that A^{-1} and B^{-1} are similar.

5. Let $\alpha = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, where $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. The reflection across the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ is the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$T(\mathbf{v}) = 2\left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} + 2\left(\frac{\mathbf{v} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right)\mathbf{b} - \mathbf{v}.$$

(a) Find $[T]_\alpha^\alpha$.

(b) Find $[I]_\alpha^\beta$ and $[I]_\beta^\alpha$, where β is the standard basis for $\mathbf{R}^3$.

(c) Find $[T]_\beta^\beta$.

6. Find the determinant of each matrix. Is either matrix invertible?

(a) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 17 & 9 & 8 & -3 \\ 2 & 4 & 6 & 8 \\ 15 & -8 & 7 & 19 \end{bmatrix}$

(b) $B = \begin{bmatrix} 5 & 1 & 2 & 5 \\ 1 & 6 & 2 & 0 \\ 7 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{bmatrix}$

7. Let A and B be $n \times n$ matrices.

(a) Show that if AB is an invertible matrix, then A and B must both be invertible.

(b) Show by example that $A + B$ could be invertible even if neither A nor B is invertible.