College of the Holy Cross, Fall 2018
Math 244, Midterm 3 Solutions

1. Let $A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 0 & 3 \\ 7 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$.

   (a) Which of the following matrix products are defined? Compute those that are.

   (i) $AB = \begin{bmatrix} 36 & 3 \\ -1 & 7 \end{bmatrix}$

   (ii) AC is undefined

   (iii) $BA = \begin{bmatrix} 4 & 12 & 18 \\ -6 & 12 & 3 \\ 16 & 3 & 27 \end{bmatrix}$

   (iv) $BC = \begin{bmatrix} 6 & 18 \\ -3 & 15 \\ 15 & 9 \end{bmatrix}$

   (v) $CA = \begin{bmatrix} 0 & 10 & 10 \\ -12 & 19 & 1 \end{bmatrix}$

   (vi) CB is undefined

   (b) Let

$$S(x_1, x_2, x_3) = (2x_1 + x_2 + 4x_3, -2x_1 + 4x_2 + x_3)$$
$$T(y_1, y_2) = (4y_1 + 2y_2, 3y_2, 7y_1 - y_2).$$

   Which of the matrices in part (a) is the matrix for $T \circ S$ with respect to the standard basis for $\mathbf{R}^3$?

   Solution. With respect to the standard basis, the matrix for S is A and the matrix for T is B, so the matrix for $T \circ S$ is BA, (iii).

2. Let $S : U \to V$ and $T : V \to W$ be linear transformations.

   (a) Prove that if S and T are surjective, then $T \circ S$ is surjective.

   Solution. Since $T \circ S$ is a mapping from U to W, we need to show that for any $\mathbf{w} \in W$ there exists some $\mathbf{u} \in U$ such that $(T \circ S)(\mathbf{u}) = \mathbf{w}$.

   Let $\mathbf{w} \in W$. Since T is surjective, there is some $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. Since S is surjective, there is some $\mathbf{u} \in U$ such that $S(\mathbf{u}) = \mathbf{v}$. Thus $(T \circ S)(\mathbf{u}) = T(S(\mathbf{u})) = T(\mathbf{v}) = \mathbf{w}$, as desired.

   (b) Prove that if $T \circ S$ is surjective, then T is surjective.

   Solution. Let $\mathbf{w} \in W$. Since $T \circ S$ is surjective, there is some $\mathbf{u} \in U$ such that $(T \circ S)(\mathbf{u}) = \mathbf{w}$. But $(T \circ S)(\mathbf{u}) = T(S(\mathbf{u}))$, so define $\mathbf{v} = S(\mathbf{u})$. Then $\mathbf{v} \in V$ and $T(\mathbf{v}) = \mathbf{w}$. This proves T is surjective.

3. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 4 & 0 \\ -2 & 1 & 1 \end{bmatrix}$.

(a) Find A^{-1}.

Solution. By the usual Gaussian elimination process,

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 8 & -2 & -2 & 1 & 0 \\ 0 & -3 & 3 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{8} & 0 \\ 0 & -3 & 3 & 2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{9}{4} & \frac{5}{4} & \frac{3}{8} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{8} & 0 \\ 0 & 0 & 1 & \frac{5}{9} & \frac{1}{6} & \frac{4}{9} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{9} & \frac{1}{6} & -\frac{2}{9} \\ 0 & 1 & 0 & -\frac{1}{9} & \frac{1}{6} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{5}{9} & \frac{1}{6} & \frac{4}{9} \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} \frac{2}{9} & \frac{1}{6} & -\frac{2}{9} \\ -\frac{1}{9} & \frac{1}{6} & \frac{1}{9} \\ \frac{5}{9} & \frac{1}{6} & \frac{4}{9} \end{bmatrix}$.

(b) Find the solution of $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Solution. $\mathbf{x} = A^{-1} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} \\ -\frac{1}{9} \\ \frac{14}{9} \end{bmatrix}$

(c) Suppose $T : \mathbf{R}^3 \to \mathbf{R}^3$ is the linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Let B be the matrix for T with respect to the standard basis. Find B.

Solution. The information given implies that $BA = C$, where $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Thus

$$B = CA^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}.$$

4. (a) Complete the following definition. The matrices A and B are **similar** if **there exists an invertible matrix C such that $B = C^{-1}AC$.**

(b) Suppose A and B are invertible matrices that are similar. Prove that A^{-1} and B^{-1} are similar.

Solution. Since A and B are similar, $B = C^{-1}AC$ for some invertible matrix C. If we left-multiply by C, we get $CB = AC$. Right-multiply by B^{-1} to get $C = ACB^{-1}$. Left-multiply by A^{-1} to get $A^{-1}C = CB^{-1}$. Finally, left multiply by C^{-1} to get $C^{-1}A^{-1}C = B^{-1}$. Therefore A^{-1} and B^{-1} are similar.

5. Let $\alpha = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, where $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. The reflection across the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ is the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$T(\mathbf{v}) = 2\left(\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} + 2\left(\frac{\mathbf{v}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} - \mathbf{v}.$$

(a) Find $[T]_\alpha^\alpha$.

Solution. Since $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c} = 0$, we have

$$\begin{aligned} T(\mathbf{a}) &= 2\mathbf{a} - \mathbf{a} = \mathbf{a} = 1\cdot\mathbf{a} + 0\cdot\mathbf{b} + 0\cdot\mathbf{c} \\ T(\mathbf{b}) &= 2\mathbf{b} - \mathbf{b} = \mathbf{b} = 0\cdot\mathbf{a} + 1\cdot\mathbf{b} + 0\cdot\mathbf{c} \\ T(\mathbf{c}) &= -\mathbf{c} = 0\cdot\mathbf{a} + 0\cdot\mathbf{b} + (-1)\cdot\mathbf{c}, \end{aligned}$$

and thus $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(b) Find $[I]_\alpha^\beta$ and $[I]_\beta^\alpha$, where β is the standard basis for $\mathbf{R}^3$.

Solution. $[I]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and $[I]_\beta^\alpha = ([I]_\alpha^\beta)^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{2}{6} \end{bmatrix}$.

(c) Find $[T]_\beta^\beta$.

Solution.

$$[T]_\beta^\beta = [I]_\alpha^\beta [T]_\alpha^\alpha [I]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{2}{6} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

6. Find the determinant of each matrix. Is either matrix invertible?

(a) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 17 & 9 & 8 & -3 \\ 2 & 4 & 6 & 8 \\ 15 & -8 & 7 & 19 \end{bmatrix}$

Solution.

$$\det(A) = 2\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 17 & 9 & 8 & -3 \\ 1 & 2 & 3 & 4 \\ 15 & -8 & 7 & 19 \end{bmatrix} = 0$$

since this matrix has a repeated row.

(b) $B = \begin{bmatrix} 5 & 1 & 2 & 5 \\ 1 & 6 & 2 & 0 \\ 7 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{bmatrix}$

Solution. Swapping rows gives

$$\det(B) = -\det\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 7 & 0 & 0 & 0 \\ 5 & 1 & 2 & 5 \end{bmatrix} = +\det\begin{bmatrix} 7 & 0 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 1 & 2 & 5 \end{bmatrix} = -\det\begin{bmatrix} 7 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 5 & 1 & 2 & 5 \end{bmatrix} = -7\cdot 4\cdot 2\cdot 5 = -280$$

7. Let A and B be $n \times n$ matrices.

(a) Show that if AB is an invertible matrix, then A and B must both be invertible.

Solution. Since AB is invertible, $\det(AB) \neq 0$. But $\det(AB) = \det(A)\det(B)$ so both $\det(A)$ and $\det(B)$ must also be nonzero, which implies that A and B are both invertible.

(b) Show by example that $A + B$ could be invertible even if neither A nor B is invertible.

Solution. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det(A) = \det(B) = 0$, so neither A nor B is invertible, but $A + B = I$ is invertible.