College of the Holy Cross, Fall 2018
Math 244, Midterm 2

1. Let
$$W_1 = \text{Span}\left(\begin{bmatrix}1\\3\\1\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right) \qquad W_2 = \text{Span}\left(\begin{bmatrix}1\\1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)$$
Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$. Prove your assertions.

2. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T(v_1, v_2) = (v_1v_2, v_1 + v_2)$. Is T a linear transformation? Prove your assertion.

3. Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation whose matrix with respect to the standard bases is $A = \begin{bmatrix}1 & 2 & 3 & 4\\ 1 & -1 & 1 & -1\\ 1 & -10 & -5 & -16\end{bmatrix}$.

   (a) Find $T(2\mathbf{e}_2 + 3\mathbf{e}_4)$.

   (b) Find bases for $\text{Ker}(T)$ and $\text{Im}(T)$.

   (c) Is T injective? Is T surjective? Explain.

   (d) Find the set of solutions of the equation $T(\mathbf{x}) = \begin{bmatrix}3\\1\\-5\end{bmatrix}$.

4. Let $\mathbf{a} = \begin{bmatrix}2\\2\\1\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}1\\-2\\2\end{bmatrix}$, and define a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ by
$$T(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}.$$
(This is the projection onto the plane spanned by $\mathbf{a}$ and $\mathbf{b}$.)

   (a) Show that T is a linear transformation.

   (b) Compute $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$ and $T(\mathbf{e}_3)$.

   (c) Find the matrix for T with respect to the standard basis for $\mathbf{R}^3$.

   (d) (Bonus) Find a basis for $\text{Ker}(T)$.

5. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V and $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W, and $T : V \to W$ is a linear transformation such that $[T]_\alpha^\beta = \begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}$.

   (a) Let $\mathbf{x} = 4\mathbf{v}_1 - 3\mathbf{v}_2$. Find the following:
      - $[\mathbf{x}]_\alpha$
      - $[T(\mathbf{x})]_\beta$
      - $T(\mathbf{x})$

(b) Find a vector $\mathbf{y} \in V$ such that $T(\mathbf{y}) = \mathbf{w}_2$.

6. Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be defined by $T(p(x)) = p''(x) + p'(x) + p(x)$.

   (a) Show that T is linear.

   (b) Find the matrix for T with respect to the basis $\alpha = \{1, x, x^2\}$.

   (c) Is T injective? Is T surjective? Prove your assertions.

7. Suppose $T : V \to W$ is a linear transformation.

   (a) Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

   (b) Let U be a subspace of W and define $Y = \{\mathbf{v} \in V : T(\mathbf{v}) \in U\}$. Show that Y is a subspace of V.

8. Let $T : \mathbf{R}^4 \to \mathbf{R}^2$ be a linear transformation.

   (a) What are the possible dimensions of $\mathrm{Ker}(T)$? Explain.

   (b) Give an example of such a transformation for which $\dim(\mathrm{Ker}(T)) = 3$.