College of the Holy Cross, Fall 2018
Math 244, Midterm 2 Solutions

1. Let

$$W_1 = \text{Span}\left(\begin{bmatrix}1\\3\\1\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right) \qquad W_2 = \text{Span}\left(\begin{bmatrix}1\\1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)$$

Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$. Prove your assertions.

Solution. Let

$$\mathbf{w}_1 = \begin{bmatrix}1\\3\\1\\3\end{bmatrix} \quad \mathbf{w}_2 = \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix} \quad \mathbf{w}_3 = \begin{bmatrix}1\\1\\2\\2\end{bmatrix} \quad \mathbf{w}_4 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Clearly $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent, so it is a basis for W_1 and thus $\dim(W_1) = 2$. Likewise, $\{\mathbf{w}_3, \mathbf{w}_4\}$ is a basis for W_2, so $\dim(W_2) = 2$.

Since

$$\text{rref}\begin{bmatrix}1 & -1 & 1 & 0\\3 & 1 & 1 & 0\\1 & -1 & 2 & 1\\3 & 1 & 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & -\frac{1}{2}\\0 & 1 & 0 & \frac{1}{2}\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

the set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent, and $\mathbf{w}_4 = -\frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2 + \mathbf{w}_3$. Thus $W_1 + W_2 = \text{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) = \text{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$. This implies $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for $W_1 + W_2$, so $\dim(W_1 + W_2) = 3$. By the dimension theorem,

$$\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) = 2 + 2 - 3 = 1.$$

2. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T(v_1, v_2) = (v_1 v_2, v_1 + v_2)$. Is T a linear transformation? Prove your assertion.

Solution. T is not linear. Let $\mathbf{v} = (1, 1)$. Then $T(\mathbf{v}) = (1, 2)$, so $2T(\mathbf{v}) = (2, 4)$. But $T(2\mathbf{v}) = T(2, 2) = (4, 4)$, so $T(2\mathbf{v}) \neq 2T(\mathbf{v})$.

3. Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation whose matrix with respect to the standard bases is $A = \begin{bmatrix}1 & 2 & 3 & 4\\1 & -1 & 1 & -1\\1 & -10 & -5 & -16\end{bmatrix}$.

   (a) Find $T(2\mathbf{e}_2 + 3\mathbf{e}_4)$.

   Solution. $T(2\mathbf{e}_2 + 3\mathbf{e}_4) = \begin{bmatrix}1 & 2 & 3 & 4\\1 & -1 & 1 & -1\\1 & -10 & -5 & -16\end{bmatrix}\begin{bmatrix}0\\2\\0\\3\end{bmatrix} = \begin{bmatrix}16\\-5\\-68\end{bmatrix}$

   (b) Find bases for $\text{Ker}(T)$ and $\text{Im}(T)$.

   Solution. Since

$$\text{rref}(A) = \begin{bmatrix}1 & 0 & \frac{5}{3} & \frac{2}{3}\\0 & 1 & \frac{2}{3} & \frac{5}{3}\\0 & 0 & 0 & 0\end{bmatrix}$$

the first two columns of A are linearly independent, and the last two are linear combinations of the first two. Thus $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -10 \end{bmatrix} \right\}$ is a basis for $\text{Im}(T)$.

From the echelon form we also see that the solution of $A\mathbf{x} = \mathbf{0}$ is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3}x_3 - \frac{2}{3}x_4 \\ -\frac{2}{3}x_3 - \frac{5}{3}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{5}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{2}{3} \\ -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix},$$

so $\left\{ \begin{bmatrix} -\frac{5}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Ker}(T)$.

(c) Is T injective? Is T surjective? Explain.

Solution. T is not injective since $\dim(\text{Ker}(T)) = 2 > 0$. T is not surjective since $\dim(\text{Im}(T)) = 2 < 3$.

(d) Find the set of solutions of the equation $T(\mathbf{x}) = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$.

Solution. Since $T(\mathbf{e}_3) = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$, $\mathbf{x}_p = \mathbf{e}_3$ is one particular solution of the equation. Thus the set of solutions is $\{\mathbf{e}_3\} + \text{Ker}(T)$. In parametric form

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{5}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{2}{3} \\ -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix}$$

4. Let $\mathbf{a} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, and define a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$T(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} + \left(\frac{\mathbf{v} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b}.$$

(This is the projection onto the plane spanned by **a** and **b**.)

(a) Show that T is a linear transformation.

Solution. Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^2$, and $c \in \mathbf{R}$. Then

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= \left(\frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} + \left(\frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b} = \left(\frac{\mathbf{u} \cdot \mathbf{a} + \mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} + \left(\frac{\mathbf{u} \cdot \mathbf{b} + \mathbf{v} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} + \left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} + \left(\frac{\mathbf{u} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b} + \left(\frac{\mathbf{v} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} + \left(\frac{\mathbf{u} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b} + \left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} + \left(\frac{\mathbf{v} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b} = T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned}
T(c\mathbf{v}) &= \left(\frac{(c\mathbf{v})\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} + \left(\frac{(c\mathbf{v})\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} \\
&= \left(\frac{c(\mathbf{v}\cdot\mathbf{a})}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} + \left(\frac{c(\mathbf{v}\cdot\mathbf{b})}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} \\
&= c\left(\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} + c\left(\frac{\mathbf{v}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} \\
&= cT(\mathbf{v})
\end{aligned}$$

so T is linear.

(b) Compute $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$ and $T(\mathbf{e}_3)$.
Solution.

$$T(\mathbf{e}_1) = \left(\frac{\mathbf{e}_1\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} + \left(\frac{\mathbf{e}_1\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = 2\mathbf{a} + \mathbf{b} = \begin{bmatrix}5\\2\\4\end{bmatrix}$$

$$T(\mathbf{e}_2) = \left(\frac{\mathbf{e}_2\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} + \left(\frac{\mathbf{e}_2\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = 2\mathbf{a} - 2\mathbf{b} = \begin{bmatrix}2\\8\\-2\end{bmatrix}$$

$$T(\mathbf{e}_3) = \left(\frac{\mathbf{e}_3\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} + \left(\frac{\mathbf{e}_3\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \mathbf{a} + 2\mathbf{b} = \begin{bmatrix}4\\-2\\5\end{bmatrix}$$

(c) Find the matrix for T with respect to the standard basis for $\mathbf{R}^3$.
Solution. The columns of the matrix for T with respect to the standard basis are the vectors found in part (b). Thus the matrix for T is $A = \begin{bmatrix}5 & 2 & 4\\2 & 8 & -2\\4 & -2 & 5\end{bmatrix}$.

(d) Find a basis for $\mathrm{Ker}(T)$.

Solution. Since $\mathrm{rref}(A) = \begin{bmatrix}1 & 0 & 1\\0 & 1 & -\frac{1}{2}\\0 & 0 & 0\end{bmatrix}$, $\mathrm{Ker}(T) = \mathrm{Span}\left(\begin{bmatrix}-1\\\frac{1}{2}\\1\end{bmatrix}\right)$.

5. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V and $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W, and $T : V \to W$ is a linear transformation such that $[T]_\alpha^\beta = \begin{bmatrix}3 & 1\\2 & 1\end{bmatrix}$.

(a) Let $\mathbf{x} = 4\mathbf{v}_1 - 3\mathbf{v}_2$. Find the following:

- $[\mathbf{x}]_\alpha$
  Solution. $[\mathbf{x}]_\alpha = \begin{bmatrix}4\\-3\end{bmatrix}$
- $[T(\mathbf{x})]_\beta$
  Solution. $[T(\mathbf{x})]_\beta = [T]_\alpha^\beta[\mathbf{x}]_\alpha = \begin{bmatrix}3 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}4\\-3\end{bmatrix} = \begin{bmatrix}9\\5\end{bmatrix}$
- $T(\mathbf{x})$
  Solution. $T(\mathbf{x}) = 9\mathbf{v}_1 + 5\mathbf{v}_2$.

(b) Find a vector $\mathbf{y} \in V$ such that $T(\mathbf{y}) = \mathbf{w}_2$.

Solution. Let $[\mathbf{y}]_\alpha = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then since $[T(\mathbf{y})]_\beta = [T]_\alpha^\beta [\mathbf{y}]_\alpha$ and $[T(\mathbf{y})]_\beta = [\mathbf{w}_2]_\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we need to solve

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The solution of this system is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, so $\mathbf{y} = -3\mathbf{v}_1 + \mathbf{v}_2$.

6. Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be defined by $T(p(x)) = p''(x) + p'(x) + p(x)$.

(a) Show that T is linear.

Solution. Let p and q be in $P_2(\mathbf{R})$ and let $c \in \mathbf{R}$. Then

$$\begin{aligned} T(p+q)(x) &= (p+q)''(x) + (p+q)'(x) + (p+q)(x) \\ &= p''(x) + q''(x) + p'(x) + q'(x) + p(x) + q(x) \\ &= p''(x) + p'(x) + p(x) + q''(x) + q'(x) + q(x) \\ &= T(p)(x) + T(q)(x) \end{aligned}$$

and

$$T(cp)(x) = (cp)''(x) + (cp)'(x) + (cp)(x) = cp''(x) + cp'(x) + cp(x) = cT(p)(x)$$

so T is linear.

(b) Find the matrix for T with respect to the basis $\alpha = \{1, x, x^2\}$.

Solution.

$$\begin{aligned} T(1) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x) &= 1 + x = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \\ T(x^2) &= 2 + 2x + x^2 = 2 \cdot 1 + 2 \cdot x + 2 \cdot x^2 \end{aligned}$$

so the matrix for T with respect to α is

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Is T injective? Is T surjective? Prove your assertions.

Solution. Suppose $p(x) = a + bx + cx^2$ is in $\text{Ker}(T)$. Then $T(p) = 0$, so $[T(p)]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

But

$$[T(p)]_\alpha = [T]_\alpha^\alpha [p]_\alpha = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+2c \\ b+2c \\ c \end{bmatrix},$$

so we have $a + b + 2c = 0$, $b + 2c = 0$ and $c = 0$, which implies $a = b = c = 0$ and thus $p = 0$. Thus $\text{Ker}(T) = \{0\}$ and T is injective. Since $V = W = P_2(\mathbf{R})$, T is also surjective.

7. Suppose $T : V \to W$ is a linear transformation.

   (a) Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

   Solution. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Then there exist scalars c_1, c_2 and c_3, not all of which are zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

   Thus by linearity of T,

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(\mathbf{0}) = \mathbf{0}$$

   so $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

   (b) Let U be a subspace of W and define $Y = \{\mathbf{v} \in V : T(\mathbf{v}) \in U\}$. Show that Y is a subspace of V.

   Solution. Let $\mathbf{y}_1, \mathbf{y}_2 \in Y$ and $c \in \mathbf{R}$. Then $T(\mathbf{y}_1) \in U$ and $T(\mathbf{y}_2) \in U$. Since U is a subspace of W, $cT(\mathbf{y}_1) + T(\mathbf{y}_2) \in U$. By linearity, $cT(\mathbf{y}_1) + T(\mathbf{y}_2) = T(c\mathbf{y}_1 + \mathbf{y}_2)$, so $T(c\mathbf{y}_1 + \mathbf{y}_2) \in U$, which by definition of Y implies $c\mathbf{y}_1 + \mathbf{y}_2 \in Y$. Thus Y is a subspace of V.

8. Let $T : \mathbf{R}^4 \to \mathbf{R}^2$ be a linear transformation.

   (a) What are the possible dimensions of $\text{Ker}(T)$? Explain.

   Solution. By the dimension theorem $\dim(\text{Im}(T)) + \dim(\text{Ker}(T)) = \dim(\mathbf{R}^4) = 4$, so $\dim(\text{Ker}(T)) \leq 4$. But $\text{Im}(T)$ is a subspace of $\mathbf{R}^2$, so $\dim(\text{Im}(T)) \leq 2$, and thus $\dim(\text{Ker}(T)) \geq 2$. Therefore $\dim(\text{Ker}(T))$ could be 2, 3, or 4.

   (b) Give an example of such a transformation for which $\dim(\text{Ker}(T)) = 3$.

   Solution. Let T be the linear transformation whose matrix with respect to the standard basis is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then $\text{Ker}(T) = \text{Span}(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$, so $\dim(\text{Ker}(T)) = 3$.