College of the Holy Cross, Fall 2018
Math 244, Linear Algebra
Midterm 2 Practice Problems

1. Let $W_1 = \text{Span}\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$ and $W_2 = \text{Span}\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}2\\4\\8\\10\end{bmatrix}\right)$. Find bases for W_1+W_2 and $W_1 \cap W_2$ and verify that their dimensions satisfy Theorem 1.6.18.

   Solution. Define

$$\mathbf{w}_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \mathbf{w}_4 = \begin{bmatrix}2\\4\\8\\10\end{bmatrix}.$$

   Then $W_1 + W_2 = \text{Span}\left(\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}\right)$. Since

$$\text{rref}\left(\begin{bmatrix}1&1&1&2\\0&1&2&4\\1&1&3&8\\0&1&4&10\end{bmatrix}\right) = \begin{bmatrix}1&0&0&1\\0&1&0&-2\\0&0&1&3\\0&0&0&0\end{bmatrix}$$

   the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent, and $\mathbf{w}_4 = \mathbf{w}_1 - 2\mathbf{w}_2 + 3\mathbf{w}_3$. Thus $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for $W_1 + W_2$.

   To find $W_1 \cup W_2$ we need to determine which linear combinations of $\mathbf{w}_1$ and $\mathbf{w_2}$ are also linear combinations of $\mathbf{w}_3$ and $\mathbf{w}_4$. That is, we need to find c_1, c_2, c_3 and c_4 such that $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 = c_3\mathbf{w}_3 + c_4\mathbf{w}_4$, or equivalently $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 - c_3\mathbf{w}_3 - c_4\mathbf{w}_4 = \mathbf{0}$. This amounts to finding the kernel of the matrix above. From the echelon form, we see that

$$\text{Ker}\left(\begin{bmatrix}1&1&1&2\\0&1&2&4\\1&1&3&8\\0&1&4&10\end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix}-1\\2\\-3\\1\end{bmatrix}\right)$$

   This implies $-\mathbf{w}_1 + 2\mathbf{w}_2 - 3\mathbf{w}_3 + \mathbf{w}_4 = \mathbf{0}$, and thus $\mathbf{w}_1 - 2\mathbf{w}_2 = -3\mathbf{w}_3 + \mathbf{w}_4$ is in $W_1 \cap W_2$, and the only other vectors in $W_1 \cap W_2$ are scalar multiples of this vector. Thus $\{\mathbf{w}_1 - 2\mathbf{w}_2\}$ is a basis for $W_1 \cap W_2$.

   Therefore we have $\dim(W_1) = \dim(W_2) = 2$, $\dim(W_1+W_2) = 3$ and $\dim(W_1 \cap W_2) = 1$, and we have $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$, in agreement with Theorem 1.6.18.

2. Suppose $\dim(V) = n$. Show that a set of n vectors in V is linearly independent if and only if it spans V.

   Solution. Let S be a set of n elements in V, and let α be any basis for V. Since $\dim(V) = n$, α has n elements.

First suppose S is linearly independent. To show that S must span V, suppose to the contrary that it does not. Then there exists some vector $\mathbf{v}$ that is not in the span of S. This implies that the set $S' = S \cup \{\mathbf{v}\}$ is a linearly independent set in V consisting of $n+1$ elements. Thus S' is a linearly independent set with more elements than the spanning set α, which contradicts Theorem 1.6.10. Therefore S must span V.

Next suppose S spans V. To prove S is linearly independent, suppose it is not. Then one of the elements of S is a linear combination of the other $n-1$ vectors. Let S' be the set consisting of these $n-1$ vectors. Then S' still spans V. Thus α is a linearly independent set with more elements than the spanning set S', again contradicting Theorem 1.6.10. Therefore S must be linearly independent.

3. For each transformation below, determine (with proof) whether or not is is linear.

   (a) $T : P(\mathbf{R}) \to P(\mathbf{R})$ defined by $T(p(x)) = xp''(x) + 5x^2p(x)$.
   Solution. Let $p_1, p_2 \in P(\mathbf{R})$ and $c \in \mathbf{R}$. Then

$$\begin{aligned} T((cp_1 + p_2)(x)) &= x(cp_1 + p_2)''(x) + 5x^2(cp_1 + p_2)(x) \\ &= x(cp_1''(x) + p_2''(x)) + 5x^2cp_1(x) + 5x^2p_2(x) \\ &= c(xp_1''(x) + 5x^2p_1(x)) + xp_2''(x) + 5x^2p_2(x) \\ &= cT(p_1(x)) + T(p_2(x)) \end{aligned}$$

   so T is linear.

   (b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(v_1, v_2) = (v_1 + v_2 + 3, 2v_1 + 3v_2)$.
   Solution. T is not linear since $T(1,1) = (5,5)$ and $2T(1,1) = (10,10)$, but $T(2(1,1)) = T(2,2) = (7,10) \neq 2T(1,1)$.

   (c) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(v_1, v_2) = (\sin(v_1 + v_2), \cos(v_1 + v_2))$.
   Solution. T is not linear, since $T(0,0) = (0,1)$, $2T(0,0) = (0,2)$, and $T(2(0,0)) = T(0,0) = (0,1)$, so $T(2(0,0)) \neq 2T(0,0)$.

   (d) $T : P_2(\mathbf{R}) \to P_3(\mathbf{R})$ defined by $T(p(x)) = \int_5^x p(t)\,dt$.
   Solution. Let $p_1, p_2 \in P_2(\mathbf{R})$ and $c \in \mathbf{R}$. Then

$$\begin{aligned} T((cp_1 + p_2)(x)) &= \int_5^x (cp_1 + p_2)(t)\,dt \\ &= \int_5^x cp_1(t) + p_2(t)\,dt \\ &= \int_5^x cp_1(t)\,dt + \int_5^x p_2(t)\,dt \\ &= c\int_5^x p_1(t)\,dt + \int_5^x p_2(t)\,dt \\ &= cT(p_1(x)) + T(p_2(x)), \end{aligned}$$

   so T is linear.

4. The vertices of a triangle are $(0,0)$, $(2,1)$ and $(1,3)$. Find the vertices of the triangle obtained by rotating the triangle about the origin through an angle of 60 degrees.

Solution. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ denote the rotation through 60 degrees ($\pi/3$ radians). Its matrix with respect to the standard basis is

$$A = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Thus $T(0,0) = (0,0)$, $T(2,1) = (1 - \frac{\sqrt{3}}{2}, \sqrt{3} - \frac{1}{2})$ and $T(1,3) = (\frac{1}{2} - \frac{3\sqrt{3}}{2}, \frac{\sqrt{3}}{2} + \frac{3}{2})$.

5. Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1$ is a scalar multiple of $\mathbf{a}$, $\mathbf{v}_2$ is perpendicular to $\mathbf{a}$, and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$.

Solution. Let $\mathbf{v}_1 = P_{\mathbf{a}}(\mathbf{v}) = \frac{4}{5}\mathbf{a} = \begin{bmatrix} \frac{8}{5} \\ \frac{4}{5} \end{bmatrix}$ and define $\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = \begin{bmatrix} -\frac{3}{5} \\ \frac{6}{5} \end{bmatrix}$. Then $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1$ is a multiple of $\mathbf{a}$, and since $\mathbf{v}_2 \cdot \mathbf{a} = 0$, $\mathbf{v}_2$ is perpendicular to $\mathbf{a}$.

6. Let $\mathbf{a} \neq \mathbf{0}$ be a fixed vector in $\mathbf{R}^2$, and define $R_{\mathbf{a}} : \mathbf{R}^2 \to \mathbf{R}^2$ by $R_{\mathbf{a}}(\mathbf{v}) = 2P_{\mathbf{a}}(\mathbf{v}) - \mathbf{v}$, where $P_{\mathbf{a}}$ is the projection onto the line spanned by $\mathbf{a}$. This is called the reflection across the line spanned by $\mathbf{a}$.

   (a) Show that $R_{\mathbf{a}}$ is a linear transformation.

   Solution. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be in $\mathbf{R}2$ and $c \in \mathbf{R}$. Then, since $P_{\mathbf{a}}$ is a linear transformation, we have

$$\begin{aligned} R_{\mathbf{a}}(c\mathbf{v}_1 + \mathbf{v}_2) &= 2P_{\mathbf{a}}(c\mathbf{v}_1 + \mathbf{v}_2) - (c\mathbf{v}_1 + \mathbf{v}_2) \\ &= 2(cP_{\mathbf{a}}(\mathbf{v}_1) + P_{\mathbf{a}}(\mathbf{v}_2)) - c\mathbf{v}_1 - \mathbf{v}_2 \\ &= c(2P_{\mathbf{a}}(\mathbf{v}_1) - \mathbf{v}_1) + 2P_{\mathbf{a}}(\mathbf{v}_2) - \mathbf{v}_2 \\ &= cR_{\mathbf{a}}(\mathbf{v}_1) + R_{\mathbf{a}}(\mathbf{v}_2) \end{aligned}$$

   and thus $R_{\mathbf{a}}$ is linear.

   (b) Find the matrix for $R_{\mathbf{a}}$ with respect to the standard basis.

   Solution. Since

$$R_{\mathbf{a}}(\mathbf{e}_1) = 2P_{\mathbf{a}}(\mathbf{e}_1) - \mathbf{e}_1 = 2\left(\frac{\mathbf{e}_1 \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} - \mathbf{e}_1 = \frac{2a_1}{a_1^2 + a_2^2}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{a_1^2 + a_2^2}\begin{bmatrix} a_1^2 - a_2^2 \\ 2a_1a_2 \end{bmatrix}$$

$$R_{\mathbf{a}}(\mathbf{e}_2) = 2P_{\mathbf{a}}(\mathbf{e}_2) - \mathbf{e}_2 = 2\left(\frac{\mathbf{e}_2 \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} - \mathbf{e}_2 = \frac{2a_2}{a_1^2 + a_2^2}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{a_1^2 + a_2^2}\begin{bmatrix} 2a_1a_2 \\ a_2^2 - a_1^2 \end{bmatrix}$$

   the matrix for $R_{\mathbf{a}}$ is $\frac{1}{a_1^2+a_2^2}\begin{bmatrix} a_1^2 - a_2^2 & 2a_1a_2 \\ 2a_1a_2 & a_2^2 - a_1^2 \end{bmatrix}$.

   (c) Let $\mathbf{b}$ be any nonzero vector that is perpendicular to $\mathbf{a}$. Find the matrix for $R_{\mathbf{a}}$ with respect to the basis $\{\mathbf{a}, \mathbf{b}\}$.

Solution. Since

$$R_{\mathbf{a}}(\mathbf{a}) = 2(\mathbf{a}) - \mathbf{a} = \mathbf{a} = 1 \cdot \mathbf{a} + 0 \cdot \mathbf{b}$$
$$R_{\mathbf{a}}(\mathbf{b}) = 2(\mathbf{0}) - \mathbf{b} = -\mathbf{b} = 0 \cdot \mathbf{a} + (-1) \cdot \mathbf{b},$$

the matrix for $R_{\mathbf{a}}$ with respect to this basis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

7. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V and $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W. Let $T : V \to W$ be a linear transformation such that $[T]_\alpha^\beta = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$. Find $T(2\mathbf{v}_1 - 3\mathbf{v}_2)$.

Solution. Let $\mathbf{v} = 2\mathbf{v}_1 - 3\mathbf{v}_2$. Then $[\mathbf{v}]_\alpha = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, so by Proposition 2.2.15

$$[T(\mathbf{v})]_\beta = [T]_\alpha^\beta [\mathbf{v}]_\alpha = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \end{bmatrix},$$

which means $T(\mathbf{v}) = 12\mathbf{w}_1 + 5\mathbf{w}_2$.

8. For each linear transformation T given, find bases for $\text{Ker}(T)$ and $\text{Im}(T)$, and determine whether the transformation is injective, surjective, both or neither.

   (a) $T : \mathbf{R}^4 \to \mathbf{R}^3$ whose matrix with respect to the standard bases is $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

   Solution. Since

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   the kernel of T is the set of solutions of

$$\begin{array}{rcrcrcl} x_1 & & - & x_3 & - & 2x_4 & = 0 \\ & x_2 & + & 2x_3 & + & 3x_4 & = 0. \end{array}$$

   Solving for the basic variables x_1 and x_2 in terms of the free variables x_3 and x_4 gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix},$$

   and therefore $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Ker}(T)$. Since the first two columns of $\text{rref}(A)$ are linearly independent, and the last two columns are linear combinations of the first two, the same is true of the columns of A, and thus $\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right\}$

is a basis for $\mathrm{Im}(T)$. Since $\mathrm{Ker}(T) \neq \{\mathbf{0}\}$, T is not injective, and since $\mathrm{Im}(T) \neq \mathbf{R}^3$, T is not surjective.

(b) $T : P_3(\mathbf{R}) \to P_3(\mathbf{R})$ defined by $T(p(x)) = x^2 p'' - 2p(x)$.

Solution. Suppose $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is in $\mathrm{Ker}(T)$. Then $T(p(x)) = 0$, so

$$x^2(6a_3x + 2a_2) - 2(a_3x^3 + a_2x^2 + a_1x + a_0) = 0$$

which implies $4a_3x^3 - 2a_1x - 2a_0 = 0$. Since the set $\{1, x, x^2, x^3\}$ is linearly independent, this implies $a_0 = a_1 = a_3 = 0$, and therefore $p(x) = a_2x^2$. Hence $\mathrm{Ker}(T) = \mathrm{Span}(\{x^2\})$. To find $\mathrm{Im}(T)$, apply T to the polynomials in the basis $\{1, x, x^2, x^3\}$ to get

$$\begin{aligned} T(1) &= -2 \\ T(x) &= -2x \\ T(x^2) &= 0 \\ T(x^3) &= 4x^3 \end{aligned}$$

and thus $\mathrm{Im}(T) = \mathrm{Span}(\{1, x, x^3\})$. Since $\mathrm{Ker}(T) \neq \{0\}$, T is not injective, and since $\mathrm{Im}(T) \neq P_3(\mathbf{R})$, T is not surjective.

9. For each given pair of vector spaces V and W, list all possible pairs of dimensions $(\dim(\mathrm{Ker}(T)), \dim(\mathrm{Im}(T)))$ that a linear transformation $T : V \to W$ could have. For each possible pair of dimensions give an example of such a linear transformation.

(a) $V = \mathbf{R}^2$ and $W = \mathbf{R}^3$

Solution. By the dimension theorem, $\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) = 2$, so there are three possibilities: $(0, 2)$, $(1, 1)$, and $(2, 0)$.

An example with $\dim(\mathrm{Ker}(T)) = 0$ and $\dim(\mathrm{Im}(T)) = 2$ is the transformation with matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

An example with $\dim(\mathrm{Ker}(T)) = \dim(\mathrm{Im}(T)) = 1$ is the transformation with matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The only example with $\dim(\mathrm{Ker}(T)) = 2$ and $\dim(\mathrm{Im}(T)) = 0$ is the transformation with matrix $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, the zero transformation.

(b) $V = \mathbf{R}^3$ and $W = \mathbf{R}^2$

Solution. Again, the dimension theorem implies $\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) = 3$. But $\dim(\mathrm{Im}(T))$ is at most two, so the possibilities are $(1, 2)$, $(2, 1)$ and $(3, 0)$.

An example with $\dim(\mathrm{Ker}(T)) = 1$ and $\dim(\mathrm{Im}(T)) = 2$ is the transformation with matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

An example with $\dim(\text{Ker}(T)) = 2$ and $\dim(\text{Im}(T)) = 1$ is the transformation with matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The only example with $\dim(\text{Ker}(T)) = 3$ and $\dim(\text{Im}(T)) = 0$ is the transformation with matrix $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the zero transformation.

(c) $V = W = \mathbf{R}^2$

Solution. The possibilities are the same as in (a). Remove the third row from the matrices in (a) to find examples of each.

10. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for V and $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W. Let $T : V \to W$ be a linear transformation and suppose $T(\mathbf{v}_1) = 2\mathbf{w}_1 + 3\mathbf{w}_2$, $T(\mathbf{v}_2) = -\mathbf{w}_1 + 4\mathbf{w}_2$, and $T(\mathbf{v}_3) = \mathbf{w}_1 + 2\mathbf{w}_2$.

(a) Find $[T]_\alpha^\beta$

Solution. By definition, the coefficients of $\mathbf{w}_1$ and $\mathbf{w}_2$ in $T(\mathbf{v}_j)$ form the j^{th} column of $[T]_\alpha^\beta$, so $[T]_\alpha^\beta = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$.

(b) Let $\mathbf{v} = 2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3$. Find $[\mathbf{v}]_\alpha$, $[T(\mathbf{v})]_\beta$ and $T(\mathbf{v})$.

Solution. By definition,

$$[\mathbf{v}]_\alpha = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

and by Proposition 2.2.15,

$$[T(\mathbf{v})]_\beta = [T]_\alpha^\beta [\mathbf{v}]_\alpha = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix},$$

so $T(\mathbf{v}) = 4\mathbf{w}_2$.

11. Let $T : P_3(\mathbf{R}) \to P_2(\mathbf{R})$ be defined by $T(p(x)) = p'(x)$.

(a) Find the matrix $[T]_\alpha^\beta$ for T with respect to the bases $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2\}$.

Solution.

$$\begin{aligned} T(1) &= 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x^2) &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \\ T(x^3) &= 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2, \end{aligned}$$

so $[T]_\alpha^\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(b) Let $p(x) = 3 + 5x + 9x^2 - 5x^3$. Find $[p(x)]_\alpha$. Find $[T(p(x))]_\beta$ by computing $[T]_\alpha^\beta [p(x)]_\alpha$. Use this to write a formula for $T(p)$, and check that this is in fact $p'(x)$.

Solution.

$$[p(x)]_\alpha = \begin{bmatrix} 3 \\ 5 \\ 9 \\ -5 \end{bmatrix} \quad \text{so} \quad [T(p(x))]_\beta = [T]_\alpha^\beta [p(x)]_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 9 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ -15 \end{bmatrix}$$

Therefore $T(p(x)) = 5 \cdot 1 + 18 \cdot x + (-15) \cdot x^2$, which is the derivative of $p(x)$.

12. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation. Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and suppose $T(\mathbf{v}_1) = 3\mathbf{v}_1$ and $T(\mathbf{v}_2) = -4\mathbf{v}_2$.

(a) Find $[T]_\alpha^\alpha$.

Solution. Since $T(\mathbf{v}_1) = 3\mathbf{v}_1 + 0\mathbf{v}_2$ the first column of $[T]_\alpha^\alpha$ is $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Since $T(\mathbf{v}_2) = 0\mathbf{v}_1 + (-4)\mathbf{v}_2$, the second column of $[T]_\alpha^\alpha$ is $\begin{bmatrix} 0 \\ -4 \end{bmatrix}$, and thus $[T]_\alpha^\alpha = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$.

(b) Find $[T]_\beta^\beta$, where β is the standard basis for $\mathbf{R}^2$.

Solution. First, we know

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{e}_1 + 2\mathbf{e}_2 \\ \mathbf{v}_2 &= \mathbf{e}_1 + 3\mathbf{e}_2. \end{aligned}$$

Solving these equations for $\mathbf{e}_1$ and $\mathbf{e}_2$ gives $\mathbf{e}_1 = 3\mathbf{v}_1 - 2\mathbf{v}_2$ and $\mathbf{e}_2 = -\mathbf{v}_1 + \mathbf{v}_2$. Therefore

$$T(\mathbf{e}_1) = T(3\mathbf{v}_1 - 2\mathbf{v}_2) = 3T(\mathbf{v}_1) - 2T(\mathbf{v}_2) = 9\mathbf{v}_1 + 8\mathbf{v}_2 = 17\mathbf{e}_1 + 42\mathbf{e}_2$$

and

$$T(\mathbf{e}_2) = T(-\mathbf{v}_1 + \mathbf{v}_2) = -T(\mathbf{v}_1) + T(\mathbf{v}_2) - 3\mathbf{v}_1 - 4\mathbf{v}_2 = -7\mathbf{e}_1 - 18\mathbf{e}_2,$$

so

$$[T]_\beta^\beta = \begin{bmatrix} 17 & -7 \\ 42 & -18 \end{bmatrix}.$$

To convince ourselves that this is correct, let's compute $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ using this matrix:

$$\begin{aligned} T(\mathbf{v}_1) &= \begin{bmatrix} 17 & -7 \\ 42 & -18 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_1 \\ T(\mathbf{v}_2) &= \begin{bmatrix} 17 & -7 \\ 42 & -18 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -12 \end{bmatrix} = -4\mathbf{v}_2 \end{aligned}$$

13. Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$ and define $T : \mathbf{R}^3 \to \mathbf{R}^3$ by $T(\mathbf{v}) = \left(\dfrac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}$. (Recall this is the projection of $\mathbf{v}$ onto the line spanned by $\mathbf{a}$.)

(a) Find the matrix for T with respect to the standard bases.

Solution. Since $T(\mathbf{e}_1) = \frac{2}{21}\mathbf{a}$, $T(\mathbf{e}_2) = \frac{1}{21}\mathbf{a}$, and $T(\mathbf{e}_3) = \frac{-4}{21}\mathbf{a}$, the matrix for T is

$$A = \frac{1}{21} \begin{bmatrix} 4 & 2 & -8 \\ 2 & 1 & -4 \\ -8 & -4 & 16 \end{bmatrix}.$$

(b) Find bases for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution. Since

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & \frac{1}{2} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

it follows that $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Ker}(T)$, and $\{\mathbf{a}\}$ is a basis for $\mathrm{Im}(T)$.

14. Suppose $\mathrm{rref}(A) = \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ and $\mathbf{b}$ is the second column of A. Find the set of all solutions of the system of equations $A\mathbf{x} = \mathbf{b}$.

Solution. Since $A\mathbf{e}_2$ is the second column of A, and we are given that this equals $\mathbf{b}$, this means that $\mathbf{x}_p = \mathbf{e}_2$ is one particular solution of $A\mathbf{x} = \mathbf{b}$. From the echelon form of A it follows that the set of solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x}_h = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

so the general solution of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

15. Find the general solution of $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 0 \\ -1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$.

Solution.

$$\text{rref}\left(\left[\begin{array}{ccc|c} 2 & 0 & -1 & -1 \\ 1 & 2 & 0 & 2 \\ -1 & 2 & 1 & 3 \\ 1 & 2 & 0 & 2 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{4} & \frac{5}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

the general solution is

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}x_3 \\ \frac{5}{4} - \frac{1}{4}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{4} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ 1 \end{bmatrix}.$$

(Note that there are many alternate ways to write the general solution. The first vector can be any particular solution of the system, and the second vector can be scaled by any nonzero real number.)

16. Let $T : V \to W$ be a linear transformation.

(a) Suppose $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.

Solution. Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. Then by linearity,

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(\mathbf{0}) = \mathbf{0},$$

so since $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent it follows that $c_1 = c_2 = c_3 = 0$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

(b) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, and T is injective. Show that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent.

Solution. Suppose $c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0}$. Then by linearity of T,

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0},$$

so $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ is in $\text{Ker}(T)$. But since T is injective, $\text{Ker}(T) = \{\mathbf{0}\}$, and thus $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, it now follows that $c_1 = c_2 = c_3 = 0$, so $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent.