College of the Holy Cross, Fall 2018
Math 244, Linear Algebra
Midterm 2 Practice Problems

1. Let $W_1 = \text{Span}\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$ and $W_2 = \text{Span}\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}2\\4\\8\\10\end{bmatrix}\right)$. Find bases for $W_1 + W_2$ and $W_1 \cap W_2$ and verify that their dimensions satisfy Theorem 1.6.18.

2. Suppose $\dim(V) = n$. Show that a set of n vectors in V is linearly independent if and only if it spans V.

3. For each transformation below, determine (with proof) whether or not is is linear.

   (a) $T : P(\mathbf{R}) \to P(\mathbf{R})$ defined by $T(p(x)) = xp''(x) + 5x^2 p(x)$.

   (b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(v_1, v_2) = (v_1 + v_2 + 3, 2v_1 + 3v_2)$.

   (c) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(v_1, v_2) = (\sin(v_1 + v_2), \cos(v_1 + v_2))$.

   (d) $T : P_2(\mathbf{R}) \to P_3(\mathbf{R})$ defined by $T(p(x)) = \int_5^x p(t)\,dt$.

4. The vertices of a triangle are $(0, 0)$, $(2, 1)$ and $(1, 3)$. Find the vertices of the triangle obtained by rotating the triangle about the origin through an angle of 60 degrees.

5. Let $\mathbf{a} = \begin{bmatrix}2\\1\end{bmatrix}$ and let $\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix}$. Find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1$ is a scalar multiple of $\mathbf{a}$, $\mathbf{v}_2$ is perpendicular to $\mathbf{a}$, and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$.

6. Let $\mathbf{a} \neq \mathbf{0}$ be a fixed vector in $\mathbf{R}^2$, and define $R_\mathbf{a} : \mathbf{R}^2 \to \mathbf{R}^2$ by $R_\mathbf{a}(\mathbf{v}) = 2P_\mathbf{a}(\mathbf{v}) - \mathbf{v}$, where $P_\mathbf{a}$ is the projection onto the line spanned by $\mathbf{a}$. This is called the reflection across the line spanned by $\mathbf{a}$.

   (a) Show that $R_\mathbf{a}$ is a linear transformation.

   (b) Find the matrix for $R_\mathbf{a}$ with respect to the standard basis.

   (c) Let $\mathbf{b}$ be any nonzero vector that is perpendicular to $\mathbf{a}$. Find the matrix for $R_\mathbf{a}$ with respect to the basis $\{\mathbf{a}, \mathbf{b}\}$.

7. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V and $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W. Let $T : V \to W$ be a linear transformation such that $[T]_\alpha^\beta = \begin{bmatrix}3 & -2\\4 & 1\end{bmatrix}$. Find $T(2\mathbf{v}_1 - 3\mathbf{v}_2)$.

8. For each linear transformation T given, find bases for $\text{Ker}(T)$ and $\text{Im}(T)$, and determine whether the transformation is injective, surjective, both or neither.

   (a) $T : \mathbf{R}^4 \to \mathbf{R}^3$ whose matrix with respect to the standard bases is $A = \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\\9 & 10 & 11 & 12\end{bmatrix}$

(b) $T : P_3(\mathbf{R}) \to P_3(\mathbf{R})$ defined by $T(p(x)) = x^2 p'' - 2p(x)$.

9. For each given pair of vector spaces V and W, list all possible pairs of dimensions $(\dim(\mathrm{Ker}(T)), \dim(\mathrm{Im}(T)))$ that a linear transformation $T : V \to W$ could have. For each possible pair of dimensions give an example of such a linear transformation.

   (a) $V = \mathbf{R}^2$ and $W = \mathbf{R}^3$

   (b) $V = \mathbf{R}^3$ and $W = \mathbf{R}^2$

   (c) $V = W = \mathbf{R}^2$

10. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for V and $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W. Let $T : V \to W$ be a linear transformation and suppose $T(\mathbf{v}_1) = 2\mathbf{w}_1 + 3\mathbf{w}_2$, $T(\mathbf{v}_2) = -\mathbf{w}_1 + 4\mathbf{w}_2$, and $T(\mathbf{v}_3) = \mathbf{w}_1 + 2\mathbf{w}_2$.

    (a) Find $[T]_\alpha^\beta$

    (b) Let $\mathbf{v} = 2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3$. Find $[\mathbf{v}]_\alpha$, $[T(\mathbf{v})]_\beta$ and $T(\mathbf{v})$.

11. Let $T : P_3(\mathbf{R}) \to P_2(\mathbf{R})$ be defined by $T(p(x)) = p'(x)$.

    (a) Find the matrix $[T]_\alpha^\beta$ for T with respect to the bases $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2\}$.

    (b) Let $p(x) = 3 + 5x + 9x^2 - 5x^3$. Find $[p(x)]_\alpha$. Find $[T(p(x))]_\beta$ by computing $[T]_\alpha^\beta [p(x)]_\alpha$. Use this to write a formula for $T(p)$, and check that this is in fact $p'(x)$.

12. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation. Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and suppose $T(\mathbf{v}_1) = 3\mathbf{v}_1$ and $T(\mathbf{v}_2) = -4\mathbf{v}_2$.

    (a) Find $[T]_\alpha^\alpha$.

    (b) Find $[T]_\beta^\beta$, where β is the standard basis for $\mathbf{R}^2$.

13. Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$ and define $T : \mathbf{R}^3 \to \mathbf{R}^3$ by $T(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}$. (Recall this is the projection of $\mathbf{v}$ onto the line spanned by $\mathbf{a}$.)

    (a) Find the matrix for T with respect to the standard bases.

    (b) Find bases for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

14. Suppose $\mathrm{rref}(A) = \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ and $\mathbf{b}$ is the second column of A. Find the set of all solutions of the system of equations $A\mathbf{x} = \mathbf{b}$.

15. Find the general solution of $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 0 \\ -1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$.

16. Let $T : V \to W$ be a linear transformation.

(a) Suppose $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.

(b) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, and T is injective. Show that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent.