College of the Holy Cross, Fall 2018
Math 244, Midterm 1

1. Show that $\mathbf{R}^2$ is not a vector space if we define addition by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} +' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 + y_2 \\ x_1 + y_1 \end{bmatrix}$$

and use the usual scalar multiplication. State which vector space property fails and demonstrate that it fails. You only need to show that one property fails.

2. Determine whether or not each subset W is a subspace of the given vector space V. Prove your assertions.

   (a) $V = C(\mathbf{R})$, and $W = \{f \mid f'(x) + 5f(x) = 0 \text{ for all } x \in \mathbf{R}\}$

   (b) $V = \mathbf{R}^2$, $W = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mid v_1 + v_2 - 5 = 0 \right\}$

3. (a) Complete the following definition. The span of a set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is

   (b) Let $S = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \right\}$. Is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\mathrm{Span}(S)$? Prove your assertion.

   (c) Suppose $\mathbf{v}_3 \in \mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2\})$. Prove that $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2\})$.

   (d) Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 7 \\ 6 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Is $\mathrm{Span}(S) = \mathrm{Span}(T)$? Prove your assertion.

4. (a) Complete the following definition. A subset $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of a vector space V is linearly independent if

   (b) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Prove that $\{\mathbf{v}_1, \mathbf{v}_1 + 2\mathbf{v}_2, \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3\}$ is also linearly independent.

   (c) Is the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ in $\mathbf{R}^3$ linearly independent? Prove your assertion.

   (d) Is the set $S = \{\sin(x), \cos(x), \sin(2x)\}$ in $C(\mathbf{R})$ linearly independent? Prove your assertion.

5. Let W be the subspace of $\mathbf{R}^5$ consisting of all solutions of the following system.

$$\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & = & 0 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 3x_4 & + & 4x_5 & = & 0 \\ 5x_1 & + & 5x_2 & + & 7x_3 & + & 8x_4 & + & 6x_5 & = & 0 \end{array}$$

   (a) Find a basis for W.

   (b) What is the dimension of W?