College of the Holy Cross, Fall 2018
Math 244, Midterm 1 Solutions

1. Show that $\mathbf{R}^2$ is not a vector space if we define addition by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} +' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 + y_2 \\ x_1 + y_1 \end{bmatrix}$$

and use the usual scalar multiplication. State which vector space property fails and demonstrate that it fails. You only need to show that one property fails.

Solution. There are a number of properties that fail.

- The associative property fails. To prove this, it suffices to find an example of vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ such that $(\mathbf{x} +' \mathbf{y}) +' \mathbf{z} \neq \mathbf{x} +' (\mathbf{y} +' \mathbf{z})$. For example, if $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have

$$(\mathbf{x} +' \mathbf{y}) +' \mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} +' \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$\mathbf{x} +' (\mathbf{y} +' \mathbf{z}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} +' \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- There is also no additive identity. It is easy to see that $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not an additive identity, since for instance

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} +' \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To prove that no additive identity exists, suppose $\begin{bmatrix} a \\ b \end{bmatrix}$ is an additive identity. Then for every vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbf{R}^2$ we would have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} +' \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. But

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} +' \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_2 + b \\ x_1 + a \end{bmatrix}$$

so this would require $x_2 + b = x_1$ and $x_1 + a = x_2$. Since a and b are fixed, this clearly cannot hold for every x_1 and x_2. Thus there is no additive identity.

- The property $(c + d)\mathbf{x} = c\mathbf{x} +' d\mathbf{x}$ also fails. For example, if $c = d = 1$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we have

$$(c + d)\mathbf{x} = 2\mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$c\mathbf{x} +' d\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} +' \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

2. Determine whether or not each subset W is a subspace of the given vector space V. Prove your assertions.

   (a) $V = C(\mathbf{R})$, and $W = \{f \mid f'(x) + 5f(x) = 0 \text{ for all } x \in \mathbf{R}\}$
   Solution. W is a subspace of V. To prove this we need to show that if $f, g \in W$ and $c \in \mathbf{R}$, then $cf + g \in W$. So suppose $f, g \in W$ and $c \in \mathbf{R}$. Then by definition of W, $f'(x) + 5f(x) = 0$ and $g'(x) + 5g(x) = 0$ for all $x \in \mathbf{R}$. Thus

$$\begin{aligned}(cf+g)'(x) + 5(cf+g)(x) &= cf'(x) + g'(x) + 5cf(x) + 5g(x)\\ &= c(f'(x) + 5f(x)) + (g'(x) + 5g(x))\\ &= c(0) + 0\\ &= 0.\end{aligned}$$

   This proves that $cf + g \in W$, and thus W is a subspace of V.

   (b) $V = \mathbf{R}^2$, $W = \left\{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mid v_1 + v_2 - 5 = 0\right\}$
   Solution. W is **not** a subspace of V. It is not closed under addition or scalar multiplication. For instance, both $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ are in W, but their sum, $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ is not in W $(3 + 7 - 5 \neq 0)$. Also, $2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ is not in W $(4 + 6 - 5 \neq 0)$.

3. (a) Complete the following definition. The span of a set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is
   Solution. The span of S is the set of all linear combinations of vectors in S.

   (b) Let $S = \left\{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}\right\}$. Is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in Span(S)? Prove your assertion.
   Solution. We need to determine whether of not there exist scalars c_1 and c_2 such that

$$c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   This leads to the system of equations

$$\begin{array}{rcrcl} 3c_1 & + & 5c_2 & = & 1 \\ 2c_1 & - & c_2 & = & 1 \\ c_1 & + & 2c_2 & = & 1. \end{array}$$

   If we add 2 times the second equation to the third equation, we get $5c_1 = 3$, which implies $c_1 = \frac{3}{5}$. Substitution this into either the second or third equation gives $c_2 = \frac{1}{5}$. But these values of c_1 and c_2 do not satisfy the first equation. Hence the system has no solution, and therefore $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in Span(S).

(c) Suppose $\mathbf{v}_3 \in \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\})$. Prove that $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\})$.

Solution. To prove equality of two sets, we must show that each one is a subset of the other.

First suppose $\mathbf{v} \in \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\})$. Then there exist scalars c_1 and c_2 such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. But this implies $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$, so $\mathbf{v} \in \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$. This proves $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\}) \subseteq \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$.

Next suppose $\mathbf{v} \in \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$. Then there exist scalars c_1, c_2 and c_3 such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Since we are given $\mathbf{v}_3 \in \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\})$, there are scalars b_1 and b_2 such that $\mathbf{v}_3 = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$, and thus

$$\begin{aligned}\mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(b_1\mathbf{v}_1 + b_2\mathbf{v}_2) \\ &= (c_1 + c_3b_1)\mathbf{v}_1 + (c_2 + c_3b_2)\mathbf{v}_2.\end{aligned}$$

This implies $\mathbf{v}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and thus $\mathbf{v} \in \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\})$. Hence $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) \subseteq \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\})$.

(d) Let $S = \left\{ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix}7\\6\\5\\4\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}$. Is $\text{Span}(S) = \text{Span}(T)$? Prove your assertion.

Solution. The sets $\text{Span}(S)$ and $\text{Span}(T)$ are equal if and only if each one is a subset of the other. Let's denote by $\mathbf{v}_1$ and $\mathbf{v}_2$ the two vectors in S, and $\mathbf{w}_1$ and $\mathbf{w}_2$ the vectors in T. The question is asking whether or not every linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$, and vice versa.

First, notice that setting $\mathbf{v}_1 = c_1\mathbf{w}_1 + c_2\mathbf{w}_2$ and solving the resulting system of equations, it follows that $\mathbf{v}_1 = -\mathbf{w}_1 + 8\mathbf{w}_2$, so $\mathbf{v}_1$ is in $\text{Span}(T)$. Likewise, $\mathbf{v}_2$ is in $\text{Span}(T)$ since $\mathbf{v}_2 = -\mathbf{w}_1 + 12\mathbf{w}_2$. Thus, for any vector $\mathbf{v}$ in $\text{Span}(S)$, we have $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ for some a_1 and a_2, so

$$\begin{aligned}\mathbf{v} &= a_1(-\mathbf{w}_1 + 8\mathbf{w}_2) + a_2(-\mathbf{w}_1 + 12\mathbf{w}_2) \\ &= (-a_1 - a_2)\mathbf{w}_1 + (8a_1 + 12a_2)\mathbf{w}_2\end{aligned}$$

and therefore $\mathbf{v} \in \text{Span}(T)$. This shows $\text{Span}(S) \subseteq \text{Span}(T)$.

Similarly, since $\mathbf{w}_1 = -3\mathbf{v}_1 + 2\mathbf{v}_2$ and $\mathbf{w}_2 = -\frac{1}{4}\mathbf{v}_1 + \frac{1}{4}\mathbf{v}_2$, it follows that $\text{Span}(T) \subseteq \text{Span}(S)$, and thus $\text{Span}(S) = \text{Span}(T)$.

4. (a) Complete the following definition. A subset $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of a vector space V is linearly independent if

Solution. S is linearly independent if the only solution of $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ is $c_1 = \cdots = c_k = 0$.

(b) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Prove that $\{\mathbf{v}_1, \mathbf{v}_1 + 2\mathbf{v}_2, \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3\}$ is also linearly independent.

Solution. Suppose $c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + 2\mathbf{v}_2) + c_3(\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3) = \mathbf{0}$. Then

$$(c_1 + c_2 + c_3)\mathbf{v}_1 + (2c_2 + 2c_3)\mathbf{v}_2 + 3c_3\mathbf{v}_3 = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, this implies

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & c_3 & = & 0 \\ & & 2c_2 & + & 2c_3 & = & 0 \\ & & & & 3c_3 & = & 0. \end{array}$$

The third equation implies $c_3 = 0$. Plugging this into the second equation gives $c_2 = 0$, and finally, using the first equation, we obtain $c_1 = 0$. Hence $\{\mathbf{v}_1, \mathbf{v}_1+2\mathbf{v}_2, \mathbf{v}_1+2\mathbf{v}_2+3\mathbf{v}_3\}$ is linearly independent.

(c) Is the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ in $\mathbf{R}^3$ linearly independent? Prove your assertion.

Solution. Suppose $c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This leads to the system

$$\begin{array}{ccccccc} c_1 & + & 4c_2 & + & 7c_3 & = & 0 \\ 2c_1 & + & 5c_2 & + & 8c_3 & = & 0 \\ 3c_1 & + & 6c_2 & + & 9c_3 & = & 0. \end{array}$$

The echelon form of this system is

$$\begin{array}{ccccccc} c_1 & & & - & c_3 & = & 0 \\ & & c_2 & + & 2c_3 & = & 0 \\ & & & & 0 & = & 0. \end{array}$$

The variable c_3 is a free variable. Setting $c_3 = 1$ gives $c_1 = 1$ and $c_2 = -2$. Thus the set S is not linearly independent.

(d) Is the set $S = \{\sin(x), \cos(x), \sin(2x)\}$ in $C(\mathbf{R})$ linearly independent? Prove your assertion.

Solution. Suppose $c_1 \sin(x) + c_2 \cos(x) + c_3 \sin(2x) = 0$ for all x in $\mathbf{R}$. When $x = 0$, this becomes $c_2 = 0$. When $x = \pi/2$, it becomes $c_1 = 0$. Thus $c_3 \sin(2x) = 0$ for all x, so setting $x = \pi/4$ gives $c_3 = 0$. Therefore S is linearly independent.

5. Let W be the subspace of $\mathbf{R}^5$ consisting of all solutions of the following system.

$$\begin{array}{ccccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & = & 0 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 3x_4 & + & 4x_5 & = & 0 \\ 5x_1 & + & 5x_2 & + & 7x_3 & + & 8x_4 & + & 6x_5 & = & 0 \end{array}$$

(a) Find a basis for W.

Solution. The echelon form of the system is

$$\begin{array}{ccccccccc} x_1 & & & - & x_4 & - & x_5 & = & 0 \\ & x_2 & & + & 4x_4 & - & 9x_5 & = & 0 \\ & & x_3 & - & x_4 & + & 8x_5 & = & 0. \end{array}$$

Solving for the basic variables x_1, x_2 and x_3 in terms of the free variables x_4 and x_5 gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_4 + x_5 \\ -4x_4 + 9x_5 \\ x_4 - 8x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 9 \\ -8 \\ 0 \\ 1 \end{bmatrix},$$

so $W = \text{Span}(S)$, where

$$S = \left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -8 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Neither vector in S is a scalar multiple of the other, so S is linearly independent, and therefore a basis for W.

(b) What is the dimension of W?

Solution. 2