**College of the Holy Cross, Fall 2018**
**Math 244, Homework 9 Solutions**

1. For each matrix do the following:
   - Find all real eigenvalues.
   - For each eigenvalue $\lambda$, find a basis for $E_\lambda$.
   - Determine whether or not the matrix is diagonalizable, and if it is, find an eigenbasis.

   (a) $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

   **Solution.** The characteristic polynomial of $A$ os

   $$p_A(\lambda) = \det(A - \lambda I) = (1-\lambda)(6-\lambda) - 6 = \lambda^2 - 7\lambda = \lambda(\lambda - 7)$$

   so the eigenvalues of $A$ are $\lambda = 0$ and $\lambda = 7$. Since

   $$\text{rref}(A - 0I) = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{rref}(A - 7I) = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix},$$

   the eigenspaces are

   $$E_0 = \text{Ker}(A - 0I) = \text{Span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad E_7 = \text{Ker}(A - 7I) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$

   $A$ is diagonalizable with eigenbasis $\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

   (b) $A = \begin{bmatrix} 0 & 4 \\ 9 & 0 \end{bmatrix}$

   **Solution.** $p_A(\lambda) = \lambda^2 - 36 = (\lambda - 6)(\lambda + 6)$ so the eigenvalues are $\lambda = 6$ and $\lambda = -6$.

   $$\text{rref}(A - 6I) = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad E_6 = \text{Span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$$

   $$\text{rref}(A + 6I) = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad E_{-6} = \text{Span}\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$$

   $A$ is diagonalizable with eigenbasis $\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$.

   (c) $A = \begin{bmatrix} 3 & 0 \\ 5 & 3 \end{bmatrix}$

   **Solution.** $p_A(\lambda) = (\lambda - 3)^2$, so $\lambda = 3$ is the only eigenvalue of $A$.

   $$\text{rref}(A - 3I) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad E_3 = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

   $A$ is not diagonalizable.

(d) $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

**Solution.** $p_A(\lambda) = (2-\lambda)(-1-\lambda)(3-\lambda)$ so the eigenvalues of $A$ are $\lambda = 2$, $\lambda = -1$ and $\lambda = 3$.

$$\text{rref}(A-2I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies E_2 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$\text{rref}(A+I) = \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies E_{-1} = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}\right)$$

$$\text{rref}(A-3I) = \begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \implies E_3 = \text{Span}\left(\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}\right)$$

$A$ is diagonalizable with eigenbasis $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}\right\}$.

(e) $A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

**Solution.** $p_A(\lambda) = (2-\lambda)(1-\lambda)^2$, so $\lambda = a$ and $\lambda = 2$ are the eigenvalues of $A$.

$$\text{rref}(A-I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies E_1 = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$\text{rref}(A-2I) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{11}{2} \\ 0 & 0 & 0 \end{bmatrix} \implies E_2 = \text{Span}\left(\begin{bmatrix} 1 \\ 11 \\ 2 \end{bmatrix}\right)$$

$A$ is not diagnonalizable since $\dim(E_1) + \dim(E_2) = 2 < 3$.

(f) $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

**Solution.** $p_A(\lambda) = (1+\lambda)(2-\lambda)^2$ so $\lambda = -1$ and $\lambda = 2$ are the eigenvalues of $A$.

$$\text{rref}(A+I) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies E_{-1} = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$\text{rref}(A-2I) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies E_2 = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

$A$ is diagonalizable with eigenbasis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2. Suppose $A$ and $B$ are similar matrices, so $B = C^{-1}AC$ for some matrix $C$.

   (a) Show that the characteristic polynomials of $A$ and $B$ are the same, and thus $A$ and $B$ have the same eigenvalues.

   **Solution.** Since

   $$B - \lambda I = C^{-1}AC - \lambda I = C^{-1}AC - \lambda C^{-1}IC = C^{-1}AC - \lambda C^{-1}(\lambda I)C = C^{-1}(A - \lambda I)C,$$

   we have

   $$\begin{aligned} p_B(\lambda) &= \det(B - \lambda I) = \det(C^{-1}(A - \lambda I)C) \\ &= \det(C^{-1}) \det(A - \lambda I) \det(C) \\ &= \frac{1}{\det(C)} \det(A - \lambda I) \det(C) \\ &= \det(A - \lambda I) = p_A(\lambda), \end{aligned}$$

   so the characteristics polynomials of $A$ and $B$ are the same.

   (b) Show that if $\mathbf{v}$ is an eigenvector of $B$ with eigenvalue $\lambda$, then $C\mathbf{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$.

   **Solution.** Suppose $\mathbf{v}$ is an eigenvector of $B$ with eigenvalue $\lambda$. Then $B\mathbf{v} = \lambda\mathbf{v}$. This implies $C^{-1}AC\mathbf{v} = \lambda\mathbf{v}$. If we left-multiply by $C$, we get $AC\mathbf{v} = C\lambda\mathbf{v} = \lambda C\mathbf{v}$. That is, $A(C\mathbf{v}) = \lambda(C\mathbf{v})$, so $C\mathbf{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$.

3. Let $A$ be an invertible matrix.

   (a) Show that 0 is not an eigenvalue of $A$.

   **Solution.** 0 is an eigenvalue of $A$ if and only if $\det(A) = \det(A - 0I) = 0$, which holds if and only if $A$ is not invertible. Thus if $A$ is invertible, 0 cannot be an eigenvalue of $A$.

   (b) Suppose $\mathbf{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$. Show that $\mathbf{v}$ is an eigenvector of $A^{-1}$ with eigenvalue $\lambda^{-1}$.

   **Solution.** If $\mathbf{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$, then $A\mathbf{v} = \lambda\mathbf{v}$. Multiplying both sides by $A^{-1}$ gives $\mathbf{v} = A^{-1}(\lambda\mathbf{v}) = \lambda A^{-1}\mathbf{v}$. From part (a) we know $\lambda \neq 0$ so we can multiply both sides by $\lambda^{-1}$ to get $\lambda^{-1}\mathbf{v} = A^{-1}\mathbf{v}$, so $\mathbf{v}$ is an eigenvector of $A^{-1}$ with eigenvalue $\lambda^{-1}$.

   (c) Show that $A$ is diagonalizable if and only if $A^{-1}$ is diagonalizable.

   **Solution.** Suppose $A$ is diagonalizable. Then $A$ has an eigenbasis $\alpha$. By part (b), each vector in $\alpha$ is also an eigenvector of $A^{-1}$, and thus $\alpha$ is an eigenbasis of $A^{-1}$ as well. Hence $A^{-1}$ is diagonalizable. Conversely, if $A^{-1}$ is diagonalizable, it has an eigenbasis $\alpha$, which by part (b) is also an eigenbasis of $A$, so $A$ is diagonalizable.

Another way to prove this is to observe that if $A$ is diagonalizable, then $A = CDC^{-1}$ for some matrix $C$, where $D$ is a diagonal matrix whose diagonal entries are the eigenvalues $\lambda_1, \dots, \lambda_n$ of $A$. Since these eigenvalues are all nonzero, $D$ is invertible and $D^{-1}$ is the diagonal matrix with diagonal entries $\lambda_1^{-1}, \dots, \lambda_n^{-1}$. Thus

$$A^{-1} = (CDC^{-1})^{-1} = (C^{-1})^{-1}D^{-1}C^{-1} = CD^{-1}C^{-1}$$

so $A^{-1}$ is similar to the diagonal matrix $D^{-1}$ and therefore diagonalizable. Conversely, if $A^{-1}$ is diagonalizable, then $A^{-1} = CDC^{-1}$, so $A = CD^{-1}C^{-1}$ and thus $A$ is diagonalizable.

4. Let $W$ be a subspace of $\mathbf{R}^n$.

   (a) Prove that $W^\perp$ is a also subspace of $\mathbf{R}^n$.

   **Solution.** Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be in $W^\perp$ and let $c \in \mathbf{R}$. Then $\mathbf{v}_1 \cdot \mathbf{w} = \mathbf{v}_2 \cdot \mathbf{w} = 0$ for every $\mathbf{w} \in W$. Then

   $$(c\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = c(\mathbf{v}_1 \cdot \mathbf{w}) + \mathbf{v}_2 \cdot \mathbf{w} = c0 + 0 = 0$$

   for every $\mathbf{w} \in W$, which implies $c\mathbf{v}_1 + \mathbf{v}_2 \in W^\perp$. Thus $W^\perp$ is a subspace of $\mathbf{R}^n$

   (b) Prove that $W \cap W^\perp = \{\mathbf{0}\}$

   **Solution.** Let $\mathbf{w} \in W \cap W^\perp$. Then $\mathbf{w} \cdot \mathbf{w} = 0$. But $\mathbf{w} \cdot \mathbf{w} = w_1^2 + \cdots + w_n^2$, so if $w_i \neq 0$ for some $i$ then $\mathbf{w} \cdot \mathbf{w} > 0$. Hence $w_i = 0$ for each $i$, which implies $\mathbf{w} = \mathbf{0}$.

   (c) Prove $(W^\perp)^\perp = W$.

   **Solution.** We need to show $(W^\perp)^\perp \subseteq W$ and $W \subseteq (W^\perp)^\perp$.

   First suppose $\mathbf{w} \in W$. Then for any $\mathbf{v} \in W^\perp$, $\mathbf{v} \cdot \mathbf{w} = 0$. This implies $\mathbf{w} \in (W^\perp)^\perp$ and thus $W \subseteq (W^\perp)^\perp$.

   Next suppose $\mathbf{x} \in (W^\perp)^\perp$. By the theorem proven in class, we know $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1 \in W$ and $\mathbf{x}_2 \in W^\perp$. Since $\mathbf{x} \in (W^\perp)^\perp$ and $\mathbf{x}_2 \in W^\perp$, we have $\mathbf{x} \cdot \mathbf{x}_2 = 0$. Also, since $\mathbf{x}_1 \in W$ and $\mathbf{x}_2 \in W^\perp$ we have $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$. Thus

   $$0 = \mathbf{x} \cdot \mathbf{x}_2 = \mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_2 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_2$$

   This implies $\mathbf{x}_2 = \mathbf{0}$, so $\mathbf{x} = \mathbf{x}_1 \in W$, and thus $(W^\perp)^\perp \subseteq W$.

   Alternately, instead of proving the second subset relation we could have counted dimensions. We know $\dim(W) + \dim(W^\perp) = n$ and $\dim(W^\perp) + \dim((W^\perp)^\perp) = n$, so it follows that $\dim((W^\perp)^\perp) = \dim(W)$. Thus since $W$ is a subset of $(W^\perp)^\perp$, $W$ is a subspace of $(W^\perp)^\perp$ with the same dimension as $(W^\perp)^\perp$. But we know that any subspace of a vector space $V$ with the same dimension as $V$ must equal $V$. Hence $W = (W^\perp)^\perp$.

5. Let $W_1$ and $W_2$ be subspaces of $\mathbf{R}^n$. Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$.

   **Solution.** We need to prove the two subset relations $(W_1 + W_2)^\perp \subseteq W_1^\perp \cap W_2^\perp$ and $(W_1 + W_2)^\perp \subseteq W_1^\perp \cap W_2^\perp$

First suppose $\mathbf{v} \in (W_1 + W_2)^\perp$. Then $\mathbf{v} \cdot \mathbf{x} = 0$ for any $\mathbf{x} \in W_1 + W_2$. Let $\mathbf{w}_1 \in W_1$. Then $\mathbf{w}_1 = \mathbf{w}_1 + \mathbf{0}$ is in $W_1 + W_2$, so $\mathbf{v} \cdot \mathbf{w}_1 = 0$. Thus $\mathbf{v} \in W_1^\perp$. Likewise, $\mathbf{v} \cdot \mathbf{w}_2 = 0$ for any $\mathbf{w}_2 \in W_2$ so $\mathbf{v} \in W_1^\perp \cap W_2^\perp$. Since this is true for any $\mathbf{v} \in (W_1 + W_2)^\perp$, we have $(W_1 + W_2)^\perp \subseteq W_1^\perp \cap W_2^\perp$.

Next suppose $\mathbf{v} \in W_1^\perp \cap W_2^\perp$, and let $\mathbf{x}$ be any element of $W_1 + W_2$. Then $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$ for some $\mathbf{w}_1 \in W_1$ and some $\mathbf{w}_2 \in W_2$. Because $\mathbf{v} \in W_1^\perp$ and $\mathbf{v} \in W_2^\perp$, we have $\mathbf{v} \cdot \mathbf{w}_1 = 0$ and $\mathbf{v} \cdot \mathbf{w}_2 = 0$, so $\mathbf{v} \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{w}_1 + \mathbf{v} \cdot \mathbf{w}_2 = 0 + 0 = 0$. Since this is true for any $\mathbf{x} \in W_1 + W_2$, we have $\mathbf{v} \in (W_1 + W_2)^\perp$. Since this is true for any $\mathbf{v} \in W_1^\perp \cap W_2^\perp$, it follows that c

6. Use the Gram-Schmidt process to find an orthonormal basis for the plane in $\mathbf{R}^3$ spanned by $(1, 2, 3)$ and $(2, 0, -1)$.

   **Solution.** Call the given vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. First define

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{and} \qquad W_1 = \text{Span}(\mathbf{v}_1)$$

   Then define

$$\mathbf{v}_2 = \mathbf{u}_2 - P_{W_1}(\mathbf{u}_2) = \mathbf{u}_2 - \left( \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} - \frac{-1}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 29 \\ 2 \\ -11 \end{bmatrix}$$

   The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the plane. Dividing each vector by its length produces the orthonormal basis

$$\left\{ \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{966}} \begin{bmatrix} 29 \\ 2 \\ -11 \end{bmatrix} \right\}$$

7. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by $(1, 1, 1, 1)$, $(0, 0, 1, 1)$ and $(1, 0, 1, 0)$.

   **Solution.** Call the given vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. First define

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad W_1 = \text{Span}(\mathbf{v}_1)$$

   Next let

$$\mathbf{v}_2 = \mathbf{u}_2 - P_{W_1}(\mathbf{u}_2) = \mathbf{u}_2 - \left( \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

and set $W_2 = \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Finally, let

$$\mathbf{v}_3 = \mathbf{u}_3 - P_{W_2}(\mathbf{u}_3) = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{1}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for the subspace. The vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ already have length 1, so dividing $\mathbf{v}_1$ by its length produces the orthonormal basis

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\}$$