College of the Holy Cross, Fall 2018
Math 244, Homework 8

1. Find the determinant of each matrix.

   (a) $\det \begin{bmatrix} 3 & 7 \\ 2 & -1 \end{bmatrix} = -17$

   (b) $\det \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix} = 0$

   (c) $\det \begin{bmatrix} 11 & 0 \\ 0 & 4 \end{bmatrix} = 44$

   (d) $\det \begin{bmatrix} 0 & 11 \\ 4 & 0 \end{bmatrix} = -44$

   (e) $\det \begin{bmatrix} 0 & 11 \\ 0 & 4 \end{bmatrix} = 0$

2. Find the determinant of the matrix for each transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ with respect to the standard basis.

   (a) Rot_θ: $\det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \cos^2(\theta) + \sin^2(\theta) = 1$

   (b) Ref_θ: $\det \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = -\cos^2(2\theta) - \sin^2(2\theta) = -1$

   (c) $\mathrm{Proj}_{\mathbf{a}}$: $\det \frac{1}{a_1^2+a_2^2} \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} = 0$

3. Give an example of 2×2 matrices A and B such that $\det(A + B) \neq \det(A) + \det(B)$.

   Solution. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$. Then $\det(A) + \det(B) = 0 + 0 = 0$, but $\det(A + B) = 10$.

4. Show that for any 2×2 matrix A and any $c \in \mathbf{R}$, $\det(cA) = c^2 \det(A)$.

   Solution. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then

   $$\det(cA) = \det \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} = (ca_{11})(c_{22}) - (ca_{12})(ca_{21}) = c^2(a_{11}a_{22} - a_{12}a_{21}) = c^2 \det(A)$$

5. Let A and B be 2×2 matrices. Prove that $\det(AB) = \det(A)\det(B)$.

   Solution. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Then

   $$\begin{aligned} \det(A)\det(B) &= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) \\ &= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} \end{aligned}$$

and

$$
\begin{aligned}
\det(AB) &= \det \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \\
&= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{21}b_{11} + a_{22}b_{21})(a_{11}b_{12} + a_{12}b_{22}) \\
&= \cancel{a_{11}b_{11}a_{21}b_{12}} + a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} + \cancel{a_{12}b_{21}a_{22}b_{22}} \\
&\quad - \cancel{a_{21}b_{11}a_{11}b_{12}} - a_{21}b_{11}a_{12}b_{22} - a_{22}b_{21}a_{11}b_{12} - \cancel{a_{22}b_{21}a_{12}b_{22}} \\
&= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} \\
&= \det(A)\det(B)
\end{aligned}
$$

6. Suppose A is a 2×2 invertible matrix. Show that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

   Solution. Since $AA^{-1} = I$, the result of the previous problem implies $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$.

7. Find the determinant of each matrix.

   (a) $\det \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ -3 & 3 & 4 \end{bmatrix} = 3\det \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} - 1\det \begin{bmatrix} 1 & 1 \\ -3 & 4 \end{bmatrix} + 2\det \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} = 3(1) - 1(7) + 2(6) = 8$

   (b) Expanding along the second row, $\det \begin{bmatrix} 7 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 5 \end{bmatrix} = 3\det \begin{bmatrix} 7 & 2 \\ 2 & 5 \end{bmatrix} = 3(31) = 93$

   (c) $\det \begin{bmatrix} 2 & 7 & 2 \\ 0 & 5 & 9 \\ 0 & 0 & -3 \end{bmatrix} = (2)(5)(-3) = -30$

   (d) $\det \begin{bmatrix} 0 & 0 & 3 \\ 0 & 4 & 7 \\ 5 & 7 & 9 \end{bmatrix} = 3\det \begin{bmatrix} 0 & 4 \\ 5 & 7 \end{bmatrix} = -60$

   (e)

$$
\begin{aligned}
\det \begin{bmatrix} 1 & 5 & 3 & 2 \\ 1 & 2 & 4 & 4 \\ 2 & 1 & 2 & 1 \\ 2 & -1 & 4 & 3 \end{bmatrix} &= \det \begin{bmatrix} 1 & 5 & 3 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & -9 & -4 & -3 \\ 0 & -11 & -2 & -1 \end{bmatrix} = \det \begin{bmatrix} -3 & 1 & 2 \\ -9 & -4 & -3 \\ -11 & -2 & -1 \end{bmatrix} \\
&= \det \begin{bmatrix} -3 & 1 & 2 \\ 9 & 4 & 3 \\ 11 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} -3 & 1 & 2 \\ 21 & 0 & -5 \\ 17 & 0 & -3 \end{bmatrix} = -1 \cdot \det \begin{bmatrix} 21 & -5 \\ 17 & -3 \end{bmatrix} \\
&= -22
\end{aligned}
$$

(f) $\det\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 7 & 2 & 0 & 0 & 0 \\ -3 & 6 & -1 & 0 & 0 \\ 4 & -8 & 3 & 7 & 0 \\ 9 & 6 & 3 & 7 & 3 \end{bmatrix} = (5)(2)(-1)(7)(3) = -210$

(g) $\det\begin{bmatrix} 2 & 1 & 3 & 1 & 2 \\ 6 & 2 & 11 & 5 & 3 \\ 4 & -8 & 3 & 7 & 13 \\ 2 & 1 & 3 & 1 & 2 \\ 9 & 6 & 3 & 7 & 3 \end{bmatrix} = 0$ since rows 1 and 4 are identical

8. For which real numbers x is each of the following matrices invertible?

(a) $\begin{bmatrix} 1-x & 2 \\ 3 & 5-x \end{bmatrix}$

Solution. Since $\det\begin{bmatrix} 1-x & 2 \\ 3 & 5-x \end{bmatrix} = (1-x)(5-x)-6 = x^2-6x-1$ is zero only when $x = \frac{6\pm\sqrt{40}}{2}$, the matrix is invertible for every real number x except $3 \pm \sqrt{10}$.

(b) $\begin{bmatrix} 1 & x \\ x & -1 \end{bmatrix}$

Solution. Since $\det\begin{bmatrix} 1 & x \\ x & -1 \end{bmatrix} = -1-x^2$ is always negative, this matrix is invertible for every real number x.

(c) $\begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix}$

Solution. Since $\det\begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix} = 0$ for every real number x, the matrix is not invertible for any x.

(d) $\begin{bmatrix} x & 1 & 2 \\ 1 & x & 1 \\ 2 & 1 & x \end{bmatrix}$

Solution. $\det\begin{bmatrix} x & 1 & 2 \\ 1 & x & 1 \\ 2 & 1 & x \end{bmatrix} = x(x^2-1) - 1(x-2) + 2(1-2x) = x^3-6x+4$.
Now notice that if $x = 2$ the first and third rows of the matrix are identical, which means its determinant must be zero. Therefore $x-2$ must be a factor of x^3-6x+4. Factoring, we have $x^3-6x+4 = (x-2)(x^2+2x-2)$, which is zero when $x = 2$ or $x = -1 \pm \sqrt{3}$, and thus the matrix is invertible for every x except these three values.

9. The transpose of a matrix A is the matrix A^t whose entry in row i column j is the

entry of A in row j column i. For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \text{then} \qquad A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Prove that for any 3×3 matrix A, $\det(A^t) = \det(A)$.

Solution. First expand $\det(A)$ and $\det(A^t)$ along the first rows:

$$\begin{aligned} \det(A) &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \\ \det(A^t) &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22} \end{aligned}$$

Since the first, second, third, and sixth terms in each expression are identical, and the fourth term in each expression matches the fifth in the other, the two expressions are equal.