College of the Holy Cross, Fall 2018
Math 244, Homework 7

1. Define

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 6 \\ 5 & -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & 2 \\ 1 & 1 \\ -2 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

There are 16 possible matrix products that can be formed by multiplying these matrices by one another (AA, AB, AC, AD, $\ldots$). Determine which of these are defined and compute them.

Solution.

$$AA = \begin{bmatrix} 2 & 5 \\ -10 & 7 \end{bmatrix} \qquad AB = \begin{bmatrix} 7 & -1 & 14 \\ 13 & -3 & -6 \end{bmatrix} \qquad AC \text{ is undefined} \qquad AD \text{ is undefined}$$

$$BA \text{ is undefined} \qquad BB \text{ is undefined} \qquad BC = \begin{bmatrix} -5 & 26 \\ 30 & 17 \end{bmatrix} \qquad BD = \begin{bmatrix} 11 & 7 & 9 \\ 26 & 6 & 17 \end{bmatrix}$$

$$CA = \begin{bmatrix} 10 & 13 \\ 0 & 4 \\ -12 & 10 \end{bmatrix} \qquad CB = \begin{bmatrix} 17 & -2 & 46 \\ 6 & -1 & 8 \\ 18 & -4 & -4 \end{bmatrix} \qquad CC \text{ is undefined} \qquad CD \text{ is undefined}$$

$$DA \text{ is undefined} \qquad DB \text{ is undefined} \qquad DC = \begin{bmatrix} 30 & 23 \\ 15 & 5 \\ 6 & 7 \end{bmatrix} \qquad DD = \begin{bmatrix} 30 & 9 & 18 \\ 12 & 3 & 6 \\ 8 & 3 & 4 \end{bmatrix}$$

2. Let $\mathrm{Proj}_{\mathbf{a}}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}$ denote the projection onto the line spanned by $\mathbf{a}$.

   (a) Prove that $\mathrm{Proj}_{\mathbf{a}} \circ \mathrm{Proj}_{\mathbf{a}} = \mathrm{Proj}_{\mathbf{a}}$
   Solution. Let $\mathbf{v} \in \mathbf{R}^2$ and set $\mathbf{w} = \mathrm{Proj}_{\mathbf{a}}(\mathbf{v})$. Then

$$\left(\mathrm{Proj}_{\mathbf{a}} \circ \mathrm{Proj}_{\mathbf{a}}\right)(\mathbf{v}) = \mathrm{Proj}_{\mathbf{a}}(\mathrm{Proj}_{\mathbf{a}}(\mathbf{v})) = \mathrm{Proj}_{\mathbf{a}}(\mathbf{w}) = \left(\frac{\mathbf{w} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} = \left(\frac{\left(\left(\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}\right) \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}$$

$$= \left(\frac{\left(\frac{\mathbf{v}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)(\mathbf{a} \cdot \mathbf{a})}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} = \mathrm{Proj}_{\mathbf{a}}(\mathbf{v})$$

   (b) Prove that $\mathrm{Proj}_{\mathbf{a}}$ is not invertible.
   Solution. Let $\mathbf{v}$ be any nonzero vector such that $\mathbf{v} \cdot \mathbf{a} = 0$. If $\mathbf{a} = (a_1, a_2)$, then $\mathbf{v} = (-a_2, a_1)$ is one such vector. Since $\mathrm{Proj}_{\mathbf{a}}(\mathbf{v}) = \mathbf{0}$, $\mathbf{v}$ is a nonzero vector in the kernel of $\mathrm{Proj}_{\mathbf{a}}$, and thus $\mathrm{Proj}_{\mathbf{a}}$ is not invertible.

3. Find the inverse of each matrix, or show that it is not invertible.

(a) $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

Solution. $\text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq I_2$ so A is not invertible.

(b) $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

Solution. $B^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix}$

Solution.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 4 & 9 & 0 & 1 & 0 \\ 1 & 8 & 27 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 6 & -1 & 1 & 0 \\ 0 & 6 & 24 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -1 & 0 \\ 0 & 2 & 6 & -1 & 1 & 0 \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -1 & 0 \\ 0 & 1 & 3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 3 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{array}\right]$$

Thus $C^{-1} = \begin{bmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$.

(d) $D = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Solution. Since $\text{rref}(D) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$, D is not invertible.

(e) $E = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution.

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

Thus $E^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$.

(f) $F = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Solution.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 0 & 1 & 0 & 0 & -1 \\ 0 & 2 & 3 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right]$$

Thus $F^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$.

4. Let $\mathrm{Rot}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ denote counter-clockwise rotation through angle θ. Recall that its matrix with respect to the standard basis for $\mathbf{R}^2$ is $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

   (a) Prove $\mathrm{Rot}_\theta \circ \mathrm{Rot}_\phi = \mathrm{Rot}_{\theta+\phi}$.

   Solution. Making use of the identities $\cos(\theta+\phi) - \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$ and $\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$ we see that the matrix for $\mathrm{Rot}_\theta \circ \mathrm{Rot}_\phi$ is

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) & -\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) & -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\phi) \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix},$$

   which is the matrix for $\mathrm{Rot}_{\theta+\phi}$.

   (b) Prove $(\mathrm{Rot}_\theta)^{-1} = \mathrm{Rot}_{-\theta}$.

   Solution. By the result in part (a), $\mathrm{Rot}_{-\theta} \circ \mathrm{Rot}_\theta = \mathrm{Rot}_\theta \circ \mathrm{Rot}_{-\theta} = \mathrm{Rot}_0$, but $\mathrm{Rot}_0 = I$, the identity transformation. Therefore $(\mathrm{Rot}_\theta)^{-1} = \mathrm{Rot}_{-\theta}$.

5. Let $\text{Ref}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ denote reflection across the line that makes angle θ with the positive x-axis. In the notation of the text this is $R_{\mathbf{a}}$, where $\mathbf{a} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$.

(a) Prove that the matrix for Ref_θ is $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Solution. As derived in a previous homework assignment, the matrix for reflection across the line spanned by $\mathbf{a} = (a_1, a_2)$ is

$$\frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1^2 - a_2^2 & 2a_1a_2 \\ 2a_1a_2 & a_2^2 - a_1^2 \end{bmatrix}$$

Substituting $a_1 = \cos(\theta)$ and $a_2 = \sin(\theta)$ and using the identities $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$, this becomes

$$\frac{1}{\cos^2(\theta) + \sin^2(\theta)} \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & \sin^2(\theta) - \cos^2(\theta) \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

(b) Prove that $(\text{Ref}_\theta)^{-1} = \text{Ref}_\theta$.

Solution. The matrix for $\text{Ref}_\theta \circ \text{Ref}_\theta$ is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & 0 \\ 0 & \sin^2(\theta) + \cos^2(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is the matrix for the identity transformation on $\mathbf{R}^2$, so it follows that $\text{Ref}_\theta \circ \text{Ref}_\theta = I$, so Ref_θ is its own inverse.

(c) Prove that $\text{Ref}_\theta \circ \text{Ref}_\phi$ is a *rotation* through some angle. What angle?

Solution. The matrix for $\text{Ref}_\theta \circ \text{Ref}_\phi$ is

$$\begin{aligned} \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} &= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\sin(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\theta - 2\phi) & -\sin(2\theta - 2\phi) \\ \sin(2\theta - 2\phi) & \cos(2\theta - 2\phi) \end{bmatrix} \end{aligned}$$

This is the matrix for Rot_α where $\alpha = 2\theta - 2\phi$.

(d) Let $\text{Rot}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ denote rotation through angle θ. Prove that $\text{Rot}_\theta \circ \text{Ref}_\phi$ is a reflection across some line. What line?

Solution. The matrix for $\text{Rot}_\theta \circ \text{Ref}_\phi$ is

$$\begin{aligned} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} &= \begin{bmatrix} \cos(\theta)\cos(2\phi) - \sin(\theta)\sin(2\phi) & \cos(\theta)\sin(2\phi) + \sin(\theta)\cos(2\phi) \\ \sin(\theta)\cos(2\phi) + \cos(\theta)\sin(2\phi) & \sin(\theta)\sin(2\phi) - \cos(\theta)\cos(2\phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta + 2\phi) & \sin(\theta + 2\phi) \\ \sin(\theta + 2\phi) & -\cos(\theta + 2\phi) \end{bmatrix} \end{aligned}$$

This is the matrix for Ref_α, were $\alpha = \phi + \frac{1}{2}\theta$.

(e) Prove that $\text{Ref}_\phi \circ \text{Rot}_\theta$ is a reflection across some line. What line?

Solution. The matrix for $\text{Ref}_\phi \circ \text{Rot}_\theta$ is

$$\begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(2\phi)\cos(\theta) + \sin(2\phi)\sin(\theta) & -\cos(2\phi)\sin(\theta) + \sin(2\phi)\cos(\theta) \\ \sin(2\phi)\cos(\theta) - \cos(2\phi)\sin(\theta) & -\sin(2\phi)\sin(\theta) - \cos(2\phi)\cos(\theta) \end{bmatrix}$$
$$= \begin{bmatrix} \cos(2\phi - \theta) & \sin(2\phi - \theta) \\ \sin(2\phi - \theta) & -\cos(2\phi - \theta) \end{bmatrix}$$

This is the matrix for Ref_α, were $\alpha = \phi - \frac{1}{2}\theta$.

6. Let $S : U \to V$ and $T : V \to W$ be linear transformations

(a) Prove that $\text{Im}(T \circ S) \subseteq \text{Im}(T)$. Give an example for which $\text{Im}(T \circ S) \neq \text{Im}(T)$.

Solution. Let $\mathbf{w} \in \text{Im}(T \circ S)$. Then there exists $\mathbf{u} \in U$ such that $(T \circ S)(\mathbf{u}) = \mathbf{w}$. But $(T \circ S)(\mathbf{u}) = T(S(\mathbf{u}))$, so setting $\mathbf{v} = S(\mathbf{u})$ it follows that $\mathbf{v} \in V$ and $T(\mathbf{v}) = \mathbf{w}$ so $\mathbf{w} \in \text{Im}(T)$.

As an example where the two images are not the same, let $S : \mathbf{R}^2 \to \mathbf{R}^2$ be the zero transformation, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the identity transformation. Then $T(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbf{R}^2$, so $\text{Im}(T) = \mathbf{R}^2$, but $(T \circ S)(\mathbf{u}) = T(S(\mathbf{u})) = T(\mathbf{0}) = \mathbf{0}$ for all $\mathbf{u} \in U$, so $\text{Im}(T \circ S) = \{\mathbf{0}\}$.

(b) Suppose S and T are invertible. Prove that $T \circ S$ is invertible and $(T \circ S)^{-1} = S^{-1} \circ T^{-1}$.

Solution. Let $\mathbf{u} \in U$. Then

$$(S^{-1} \circ T^{-1})((T \circ S)(\mathbf{u})) = S^{-1}(T^{-1}(T(S(\mathbf{u})))) = S^{-1}(S(\mathbf{u})) = \mathbf{u}$$

Likewise, for any $\mathbf{w} \in W$,

$$(T \circ S)((S^{-1} \circ T^{-1})(\mathbf{w})) = T(S(S^{-1}(T^{-1}(\mathbf{w})))) = T(T^{-1}(\mathbf{w})) = \mathbf{w}.$$

Thus, by definition of inverses, $S^{-1} \circ T^{-1}$ is the inverse of $T \circ S$.

7. Let $V = P_3(\mathbf{R})$ and $W = P_4(\mathbf{R})$, and define $S : V \to W$ by $S(p(x)) = \int_0^x p(t)\,dt$ and $T : W \to V$ by $T(p(x)) = p'(x)$. Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2, x^3, x^4\}$. Find $[S]_\alpha^\beta$, $[T]_\beta^\alpha$, $[S \circ T]_\beta^\beta$ and $[T \circ S]_\alpha^\alpha$.

Solution. Since

$$\begin{aligned} S(1) &= x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 \\ S(x) &= \frac{1}{2}x^2 = 0 \cdot 1 + 0 \cdot x + \frac{1}{2} \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 \\ S(x^2) &= \frac{1}{3}x^3 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + \frac{1}{3} \cdot x^3 + 0 \cdot x^4 \\ S(x^3) &= \frac{1}{4}x^4 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \frac{1}{4} \cdot x^4 \end{aligned}$$

we have

$$[S]_\alpha^\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

Since

$$\begin{aligned} T(1) &= 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ T(x) &= 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ T(x^2) &= 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ T(x^3) &= 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2 + 0\cdot x^3 \\ T(x^4) &= 4x^3 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + 4\cdot x^3 \end{aligned}$$

we have

$$[T]_\beta^\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Thus

$$[S\circ T]_\beta^\beta = [S]_\alpha^\beta [T]_\beta^\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [T\circ S]_\alpha^\alpha = [T]_\beta^\alpha [S]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

8. Let $\alpha = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\alpha' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ be bases for $\mathbf{R}^2$, where $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_1' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{u}_2' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a) Find the change of basis matrices $[I]_\alpha^{\alpha'}$ and $[I]_{\alpha'}^\alpha$.
Solution. Since

$$\operatorname{rref}\left[\begin{array}{cc|cc} 1 & 2 & 3 & -2 \\ 1 & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -3 \end{array}\right]$$

it follows that

$$[I]_{\alpha'}^\alpha = \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad [I]_\alpha^{\alpha'} = ([I]_{\alpha'}^\alpha)^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}.$$

(b) Let β be the standard basis. Find $[I]_\alpha^\beta$ and $[I]_\beta^\alpha$.
Solution.

$$[I]_\alpha^\beta = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad [I]_\beta^\alpha = ([I]_\alpha^\beta)^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}.$$

(c) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation such that $[T]_\alpha^\alpha = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$. Find $[T]_{\alpha'}^{\alpha'}$.
Solution.

$$[T]_{\alpha'}^{\alpha'} = [I]_\alpha^{\alpha'} [T]_\alpha^\alpha [I]_{\alpha'}^\alpha = \frac{1}{7}\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 96 \\ 8 & 11 \end{bmatrix}$$

(d) Find $[T]_\beta^\beta$ where T is the transformation in part (c).

Solution.

$$[T]_\beta^\beta = [I]_\alpha^\beta [T]_\alpha^\alpha [I]_\beta^\alpha = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 48 \\ 16 & 11 \end{bmatrix}$$