College of the Holy Cross, Fall 2018
Math 244, Homework 7

1. Define

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 6 \\ 5 & -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & 2 \\ 1 & 1 \\ -2 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

   There are 16 possible matrix products that can be formed by multiplying these matrices by one another (AA, AB, AC, AD, $\ldots$). Determine which of these are defined and compute them.

2. Let $\text{Proj}_{\mathbf{a}}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}$ denote the projection onto the line spanned by $\mathbf{a}$.

   (a) Prove that $\text{Proj}_{\mathbf{a}} \circ \text{Proj}_{\mathbf{a}} = \text{Proj}_{\mathbf{a}}$

   (b) Prove that $\text{Proj}_{\mathbf{a}}$ is not invertible.

3. Find the inverse of each matrix, or show that it is not invertible.

   (a) $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix}$

   (d) $D = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

   (e) $E = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

   (f) $F = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

4. Let $\text{Rot}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ denote counter-clockwise rotation through angle θ. Recall that its matrix with respect to the standard basis for $\mathbf{R}^2$ is $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

   (a) Prove $\text{Rot}_\theta \circ \text{Rot}_\phi = \text{Rot}_{\theta+\phi}$.

   (b) Prove $(\text{Rot}_\theta)^{-1} = \text{Rot}_{-\theta}$.

5. Let $\text{Ref}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ denote reflection across the line that makes angle θ with the positive x-axis. In the notation of the text this is $R_{\mathbf{a}}$, where $\mathbf{a} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$.

   (a) Prove that the matrix for Ref_θ is $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

   (b) Prove that $(\text{Ref}_\theta)^{-1} = \text{Ref}_\theta$.

   (c) Prove that $\text{Ref}_\theta \circ \text{Ref}_\phi$ is a *rotation* through some angle. What angle?

   (d) Let $\text{Rot}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ denote rotation through angle θ. Prove that $\text{Rot}_\theta \circ \text{Ref}_\phi$ is a reflection across some line. What line?

(e) Prove that $\text{Ref}_\phi \circ \text{Rot}_\theta$ is a reflection across some line. What line?

6. Let $S : U \to V$ and $T : V \to W$ be linear transformations

   (a) Prove that $\text{Im}(T \circ S) \subseteq \text{Im}(T)$. Give an example for which $\text{Im}(T \circ S) \neq \text{Im}(T)$.

   (b) Suppose S and T are invertible. Prove that $T \circ S$ is invertible and $(T \circ S)^{-1} = S^{-1} \circ T^{-1}$.

7. Let $V = P_3(\mathbf{R})$ and $W = P_4(\mathbf{R})$, and define $S : V \to W$ by $S(p(x)) = \int_0^x p(t)\,dt$ and $T : W \to V$ by $T(p(x)) = p'(x)$. Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2, x^3, x^4\}$. Find $[S]_\alpha^\beta$, $[T]_\beta^\alpha$, $[S \circ T]_\beta^\beta$ and $[T \circ S]_\alpha^\alpha$.

8. Let $\alpha = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\alpha' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ be bases for $\mathbf{R}^2$, where $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_1' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{u}_2' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

   (a) Find the change of basis matrices $[I]_\alpha^{\alpha'}$ and $[I]_{\alpha'}^\alpha$.

   (b) Let β be the standard basis. Find $[I]_\alpha^\beta$ and $[I]_\beta^\alpha$.

   (c) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation such that $[T]_\alpha^\alpha = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$. Find $[T]_{\alpha'}^{\alpha'}$.

   (d) Find $[T]_\beta^\beta$ where T is the transformation in part (c).