College of the Holy Cross, Fall 2018
Math 244
Solutions to Midterm 3 Practice Problems

1. Let $S : U \to V$ and $T : V \to W$ be linear transformations.

   (a) Prove that if S and T are injective, then $T \circ S$ is injective.
   Solution. Let $\mathbf{u} \in \mathrm{Ker}(T \circ S)$. Then $(T \circ S)(\mathbf{u}) = \mathbf{0}$, so $T(S(\mathbf{u})) = \mathbf{0}$, and therefore $S(\mathbf{u}) \in \mathrm{Ker}(T)$. But T is injective, so $\mathrm{Ker}(T) = \{\mathbf{0}\}$ and thus $S(\mathbf{u}) = \mathbf{0}$. This implies $\mathbf{u} \in \mathrm{Ker}(S)$. Since S is injective, $\mathrm{Ker}(S) = \{\mathbf{0}\}$, so $\mathbf{u} = \mathbf{0}$. Thus $\mathrm{Ker}(T \circ S) = \{\mathbf{0}\}$, so $T \circ S$ is injective.

   (b) Suppose $T \circ S$ is injective.

      (i) Prove S must be injective.
      Solution. Suppose $\mathbf{u} \in \mathrm{Ker}(S)$. Then $S(\mathbf{u}) = \mathbf{0}$, which implies $(T \circ S)(\mathbf{u}) = T(S(\mathbf{u})) = T(\mathbf{0}) = \mathbf{0}$. Thus $\mathbf{u} \in \mathrm{Ker}(T \circ S)$. But since $T \circ S$ is injective, $\mathrm{Ker}(T \circ S) = \{\mathbf{0}\}$ and thus $\mathbf{u} = \mathbf{0}$. Hence $\mathrm{Ker}(S) = \{\mathbf{0}\}$, so S is injective.

      (ii) Show by example that T need not be injective.
      Solution. Let $S : \mathbf{R}^2 \to \mathbf{R}^3$ have matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and let $T : \mathbf{R}^3 \to \mathbf{R}^2$ have matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then $T \circ S : \mathbf{R}^2 \to \mathbf{R}^2$ has matrix $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $T \circ S$ is injective, but T is not injective since $\mathrm{Ker}(T) = \mathrm{Span}(\mathbf{e}_3)$.

2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$.

   (a) Find the inverse of A.
   Solution.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & -2 & 3 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \end{array}\right]$$

   So $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$

   (b) Find the solution of the system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ & & x_2 & + & x_3 & = & -1 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 4 \end{array}$$

Solution. The system can be written $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$. The solution is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{1}{4} \\ -\frac{3}{4} \end{bmatrix}.$$

3. Suppose A and B are $n \times n$ invertible matrices.

   (a) Prove that AB and BA are invertible.
   Solution 1. Since $\det(A) \neq 0$ and $\det(B) \neq 0$, $\det(AB) = \det(BA) = \det(A)\det(B) \neq 0$, so both AB and BA are invertible.
   Solution 2. Since

   $$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

   and

   $$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I,$$

   the matrix $B^{-1}A^{-1}$ is the inverse of AB. Likewise $A^{-1}B^{-1}$ is the inverse of BA.

   (b) Show by example that $A + B$ is not necessarily invertible.
   Solution. Let $A = I$ and $B = -I$. Then A and B are both invertible, but $A + B = O$ (the zero matrix) is not invertible.

4. Let $T : V \to W$ be an isomorphism.

   (a) Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent if and only if $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is linearly independent.
   Solution. First suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, and assume that

   $$c_1T(\mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k) = \mathbf{0}$$

   for some scalars $c_1, \ldots, c_k$. By linearity of T this implies

   $$T(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) = \mathbf{0}.$$

   Since T is an isomorphism, it is invertible. Applying T^{-1} to both sides of this equation gives

   $$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = T^{-1}(\mathbf{0}) = \mathbf{0}.$$

   Thus, because $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, $c_1 = \cdots = c_k = 0$, which implies $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is linearly independent.
   Now suppose $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is linearly independent, and assume

   $$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

   for some scalars $c_1, \ldots, c_k$. Applying T to both sides of the equation and using linearity implies

   $$c_1T(\mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k) = \mathbf{0}.$$

   Thus, because $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is linearly independent, $c_1 = \cdots = c_k = 0$, which implies $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

(b) Suppose U is a subspace of V with $\dim(U) = k$. Prove that

$$T(U) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in U\}$$

is a subspace of W with $\dim(T(U)) = k$.

Solution. Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for U. Then α is linearly independent, so by the previous result, $\beta = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is also linearly independent.

We claim that $T(U) = \mathrm{Span}(\beta)$. To see that $\mathrm{Span}(\beta) \subseteq T(U)$, let $\mathbf{w} \in \mathrm{Span}(\beta)$. Then $\mathbf{w} = c_1T(\mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k)$ for some scalars $c_1, \ldots, c_k$. By linearity, this implies $\mathbf{w} = T(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k)$. But since α is a basis for U, $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \in U$ and thus $\mathbf{w} \in T(U)$. To prove the reverse inclusion, let $\mathbf{w} \in T(U)$. Then $\mathbf{w} = T(\mathbf{u})$ for some $\mathbf{u} \in U$. Since α is a basis for U, $\mathbf{u} = \mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k$ for some scalars $c_1, \ldots, c_k$. Thus, by linearity, $\mathbf{w} = c_1T(\mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k)$, which implies $\mathbf{w} \in \mathrm{Span}(\beta)$. This proves $T(U) = \mathrm{Span}(\beta)$.

Thus β is a basis for $T(U)$. Since β has k elements, $\dim(T(U)) = k$.

5. Let A be an $n \times n$ invertible matrix. Prove that its inverse is unique.

Solution. Suppose B and C are both inverses of A. Then $AB = BA = AC = CA = I$. Consider the matrix BAC. On the one hand, $BAC = IC = C$, while on the other hand $BAC = BI = B$. Thus $B = C$.

6. (a) Suppose A is a matrix whose columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero and orthogonal to each other. Prove that A is invertible and that A^{-1} is the matrix whose *rows* are $\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|^2}, \ldots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|^2}$.

Solution. Let B be the matrix whose rows are $\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|^2}, \ldots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|^2}$. Then

$$BA = \begin{bmatrix} \frac{\mathbf{v}_1\cdot\mathbf{v}_1}{\|\mathbf{v}_1\|^2} & \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{\|\mathbf{v}_1\|^2} & \cdots & \frac{\mathbf{v}_1\cdot\mathbf{v}_n}{\|\mathbf{v}_1\|^2} \\ \frac{\mathbf{v}_2\cdot\mathbf{v}_1}{\|\mathbf{v}_2\|^2} & \frac{\mathbf{v}_2\cdot\mathbf{v}_2}{\|\mathbf{v}_2\|^2} & \cdots & \frac{\mathbf{v}_2\cdot\mathbf{v}_n}{\|\mathbf{v}_2\|^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_n\cdot\mathbf{v}_1}{\|\mathbf{v}_n\|^2} & \frac{\mathbf{v}_n\cdot\mathbf{v}_2}{\|\mathbf{v}_n\|^2} & \cdots & \frac{\mathbf{v}_n\cdot\mathbf{v}_n}{\|\mathbf{v}_n\|^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

so $B = A^{-1}$.

(b) Use the result of part (a) to find the inverse of that matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

Solution. The columns of A are orthogonal to one another, so

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{2}{6} \end{bmatrix}$$

7. Suppose A and B are similar matrices. Prove A^n is similar to B^n for any positive integer n.

Solution. Since A and B are similar, there exists an invertible matrix Q such that $B = Q^{-1}AQ$. We claim that $B^n = Q^{-1}A^nQ$ for any positive integer n, from which the similarity of A^n and B^n follows. Let's prove this by induction. The case $n = 1$ holds by assumption. Now suppose the claim holds for some $n \geq 1$. Then $B^n = Q^{-1}A^nQ$ so using the fact that $B = Q^{-1}AQ$ we have

$$\begin{aligned} B^{n+1} &= B^nB = (Q^{-1}A^nQ)(Q^{-1}AQ) = (Q^{-1}A^n)(QQ^{-1})(AQ) = (Q^{-1}A^n)I(AQ) \\ &= (Q^{-1}A^n)(AQ) = Q^{-1}A^{n+1}Q \end{aligned}$$

so the claim holds with exponent $n+1$. By induction, the claim is true for all positive integers.

8. Prove that similarity of matrices is an equivalence relation. That is, prove the following three statements:

   - Any matrix A is similar to itself
   - If A is similar to B, then B is similar to A
   - If A is similar to B and B is similar to C, then A is similar to C.

   Solution. To prove A is similar to itself, let $Q = I$. Then $Q^{-1} = I$, so $Q^{-1}AQ = IAI = IA = A$.

   Next suppose A is similar to B. Then $B = Q^{-1}AQ$ for some matrix Q. Left-multiplying both sides by Q gives $QB = AQ$. Right-multiplying both sides by Q^{-1} gives $QBQ^{-1} = A$. Let $R = Q^{-1}$. Then $R^{-1} = Q$, so we can write the previous equation as $A = R^{-1}BR$, so B is similar to A.

   Finally, if A is similar to B and B is similar to C, there exist matrices Q and R such that $B = Q^{-1}AQ$ and $C = R^{-1}BR$. This implies $C = R^{-1}Q^{-1}AQR$. Define $S = QR$. Then $S^{-1} = R^{-1}Q^{-1}$, so the previous equation becomes $C = S^{-1}AS$, and therefore A is similar to C.

9. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that satisfies $T(2,-1,3) = (2,0,0)$ $T(7,0,7) = (0,0,-7)$ and $T(0,-3,6) = (0,3,0)$. Find the matrix for T^{-1} with respect to the standard basis.

   Solution. The given relations can be rewritten as $T^{-1}(2,0,0) = (2,-1,3)$, $T^{-1}(0,0,-7) = (7,0,7)$ and $T^{-1}(0,3,0) = (0,-3,6)$. By linearity, $T(1,0,0) = (1,-\frac{1}{2},\frac{3}{2})$, $T(0,0,1) = (-1,0,-1)$ and $T(0,1,0) = (0,-1,2)$, and thus the matrix for T with respect to the standard basis is $\begin{bmatrix} 1 & 0 & -1 \\ -\frac{1}{2} & -1 & 0 \\ \frac{3}{2} & 2 & -1 \end{bmatrix}$.

10. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation such that $T(1,2) = (4,1)$ and $T(3,-1) = (2,-1)$. Find the matrix for T with respect to the standard basis.

    Solution. Let A be the matrix for T with respect to the standard basis, and let

    $$B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix}.$$

The $AB = C$, so

$$A = CB^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 8 & 10 \\ -1 & 4 \end{bmatrix}.$$

11. Let $\alpha = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, where $\mathbf{a} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$. Recall that the projection onto the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ is the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$T(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} + \left(\frac{\mathbf{v} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b}.$$

(a) Verify that α is a basis for $\mathbf{R}^3$ and find $[I]_\alpha^\beta$ and $[I]_\beta^\alpha$ where β is the standard basis for $\mathbf{R}^3$.

Solution. The matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ whose columns are the vectors in α is invertible by the result of problem 6 since its columns are nonzero and orthogonal to one another. Thus α is linearly independent, and therefore a basis for $\mathbf{R}^3$. Since the columns of A are simply the vectors in α written in standard coordinates, $A = [I]_\alpha^\beta$. Applying the result of problem 6 again, its inverse is

$$[I]_\beta^\alpha = A^{-1} = \frac{1}{9} \begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{bmatrix}.$$

(b) Find $[T]_\alpha^\alpha$.

Solution. Since $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} = 0$, it follows from the definition of T that

$$T(\mathbf{a}) = \mathbf{a} = 1\mathbf{a} + 0\mathbf{b} + 0\mathbf{c}$$
$$T(\mathbf{b}) = \mathbf{b} = 0\mathbf{a} + 1\mathbf{b} + 0\mathbf{c}$$
$$T(\mathbf{c}) = \mathbf{0} = 0\mathbf{a} + 0\mathbf{b} + 0\mathbf{c},$$

and thus the matrix for T with respect to α is

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Find $[T]_\beta^\beta$.

$$\begin{aligned}
[T]_\beta^\beta &= [I]_\alpha^\beta [T]_\alpha^\alpha [I]_\beta^\alpha \\
&= \begin{bmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{bmatrix} \\
&= \frac{1}{9} \begin{bmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \frac{1}{9} \begin{bmatrix} 5 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 5 \end{bmatrix}
\end{aligned}$$

12. (a) Find the determinant of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 3 \\ 1 & 3 & 3 & 1 \end{bmatrix}$

Solution.

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 3 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix} = -8$$

(b) For which x is the matrix $\begin{bmatrix} x & 1 & 2 \\ 0 & x & 0 \\ 3 & 4 & x \end{bmatrix}$ invertible?

Solution. Expanding along the second row,

$$\det \begin{bmatrix} x & 1 & 2 \\ 0 & x & 0 \\ 3 & 4 & x \end{bmatrix} = x \det \begin{bmatrix} x & 2 \\ 3 & x \end{bmatrix} = x(x^2 - 6).$$

The matrix is invertible when its determinant is nonzero, which is the case for all real x except $x = 0$, $x = \sqrt{6}$ and $x = -\sqrt{6}$.