College of the Holy Cross, Fall 2018
Math 244, Midterm 3 Information

General Information

- The exam will be held on Thursday, November 29, from 5:30pm to 7:00pm, in Smith Labs 154. Plan to arrive a few minutes early to allow time to distribute the exams.
- The exam will cover material from sections 2.5, 2.6, 2.7, 3.1, 3.2, and 3.3 in the text.
- Cell-phones should be turned OFF for the duration of the exam. You may use a non-graphing, scientific calculator during the exam. No other calculator or electronic device may be used during the exam. This is a closed-book exam. No books or notes may be used during the exam.
- You will be expected to show all of your work. A correct answer with insufficient justification may not receive full credit.

Topics

- **Composition of Linear Transformations.** Know how to find the matrix for a composition of two linear transformations.
- **Inverses of Linear Transformations.** Know the definition of the inverse of a linear transformation, and be able to determine whether or not a given linear transformation is invertible. Know how to find the matrix for the inverse of a linear transformation.
- **Change of Basis.** Given two bases α and α' for a vector space V, know how to find the change of basis matrices $[I]_{\alpha}^{\alpha'}$ and $[I]_{\alpha'}^{\alpha}$. Given a linear transformation $T : V \to W$ and bases β and β' for W, know how to find $[T]_{\alpha'}^{\beta'}$ in terms of $[T]_{\alpha}^{\beta}$.
- **Determinants.** Know how to find the determinant of any matrix. Know the properties of the determinant as a function of the rows of a matrix. Know the geometric interpretation of the determinant of a 2×2 matrix. Know the formulas for determinants of products, inverses and transposes.

Important Definitions/Theorems/Axioms

I will expect you to know and be able to use all of the definitions, examples and theorems below to prove results similar to those on the homework assignments. **You should know precise statements of the definitions and theorems in bold.**

- Definition 2.5.10 and Propositions 2.5.1, 2.5.4, 2.5.9, 2.5.10, 2.5.13 and 2.5.14
- Definitions 2.6.3 and **2.6.4** and Proposition 2.6.1, 2.6.7 and 2.6.11
- Definition **2.7.6**, Proposition 2.7.3, and Theorem 2.7.5
- Definition 3.1.5 and Propositions 3.1.3, 3.1.3, and 3.1.6
- Definitions **3.2.1**, **3.2.2**, 3.2.4, 3.2.7, Lemmas 3.2.3 and 3.2.12, Propositions 3.2.11, and Theorems 3.2.8 and 3.2.14
- Definition 3.3.9, Propositions 3.3.4, 3.3.7, 3.3.11, and Corollary 3.3.8

Practice Problems

1. Let $S : U \to V$ and $T : V \to W$ be linear transformations.
   (a) Prove that if S and T are injective, then $T \circ S$ is injective.
   (b) Suppose $T \circ S$ is injective.
      (i) Prove S must be injective.
      (ii) Show by example that T need not be injective.

2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$.
   (a) Find the inverse of A.
   (b) Find the solution of the system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ & & x_2 & + & x_3 & = & -1 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 4 \end{array}$$

3. Suppose A and B are $n \times n$ invertible matrices.
   (a) Prove that AB and BA are invertible.
   (b) Show by example that $A + B$ is not necessarily invertible.

4. Let $T : V \to W$ be an isomorphism.
   (a) Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent if and only if $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is linearly independent.
   (b) Suppose U is a subspace of V with $\dim(U) = k$. Prove that

$$T(U) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in U\}$$

   is a subspace of W with $\dim(T(U)) = k$.

5. Let A be an $n \times n$ invertible matrix. Prove that its inverse is unique.

6. (a) Suppose A is a matrix whose columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthogonal to each other. Prove Suppose A is a matrix whose columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero and orthogonal to each other. Prove that A is invertible and that A^{-1} is the matrix whose *rows* are $\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|^2}, \ldots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|^2}$.
   (b) Use the result of part (a) to find the inverse of that matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

7. Suppose A and B are similar matrices. Prove A^n is similar to B^n for any positive integer n.

8. Prove that similarity of matrices is an equivalence relation. That is, prove the following three statements:

   - Any matrix A is similar to itself
   - If A is similar to B, then B is similar to A
   - If A is similar to B and B is similar to C, then A is similar to C.

9. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that satisfies $T(2, -1, 3) = (2, 0, 0)$ $T(7, 0, 7) = (0, 0, -7)$ and $T(0, -3, 6) = (0, 3, 0)$. Find the matrix for T^{-1} with respect to the standard basis.

10. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation such that $T(1, 2) = (4, 1)$ and $T(3, -1) = (2, -1)$. Find the matrix for T with respect to the standard basis.

11. Let $\alpha = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, where $\mathbf{a} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$. Recall that the projection onto the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ is the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$T(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} + \left(\frac{\mathbf{v} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b}.$$

   (a) Verify that α is a basis for $\mathbf{R}^3$ and find $[I]_\alpha^\beta$ and $[I]_\beta^\alpha$ where β is the standard basis for $\mathbf{R}^3$.

   (b) Find $[T]_\alpha^\alpha$.

   (c) Find $[T]_\beta^\beta$.

12. (a) Find the determinant of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 3 \\ 1 & 3 & 3 & 1 \end{bmatrix}$

   (b) For which x is the matrix $\begin{bmatrix} x & 1 & 2 \\ 0 & x & 0 \\ 3 & 4 & x \end{bmatrix}$ invertible?