Math 136: Calculus 2
Spring 2018
Professor Levandosky
Geometric Series Exercises

Recall the following basic facts about geometric series.

- The sum of a *finite* geometric series is

$$\sum_{n=0}^{N} cr^n = c + cr + cr^2 + \cdots + cr^N = \frac{c(1 - r^{N+1})}{1 - r} \qquad (r \neq 1)$$

- The sum of an *infinite* geometric series is

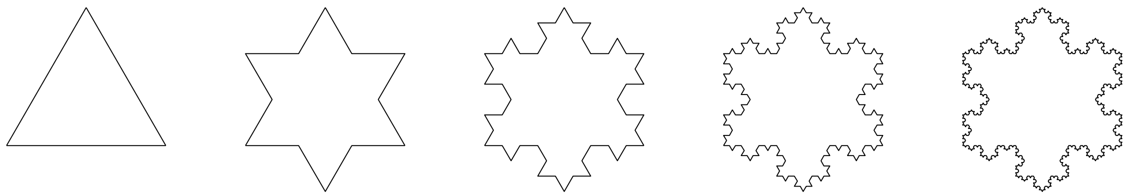
$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \cdots = \frac{c}{1 - r}$$

  as long as $-1 < r < 1$. Otherwise the series diverges.

1. Suppose you take out a loan of some amount P at an annual interest rate r compounded monthly, and that you make payments of A dollars per month.
   (a) After 1 month, the balance left to pay off is $B_1 = \left(1 + \frac{r}{12}\right) P - A$ (Why?), after 2 months, the balance is $B_2 = \left(1 + \frac{r}{12}\right)^2 P - \left(1 + \frac{r}{12}\right) A - A$. What is the balance B_3 after 3 months? What is the balance B_m after m months? Use the formula for a finite geometric series to simplify this.
   (b) Suppose you want to pay off the loan in Y years. What should the monthly payment A be in terms of P, r and Y? (Hint: Set $B_{12Y} = 0$ and solve for A.)
   (c) Apply your formula to find the monthly payment on a 10 year loan of \$10,000 with interest rate 5%. How much money do you actually wind up paying during the 10 year period?

2. A tortoise challenges Achilles to a race. Achilles can run 10 times faster than the tortoise, so the tortoise is given a head start, say 100 meters. In the time it takes Achilles to reach the tortoise's starting position, the tortoise will have gone another 10 meters. By the time Achilles travels this extra 10 meters, the tortoise will have traveled an additional 1 meter. By the time Achilles travels this additional meter, the tortoise will have traveled an additional 10 centimeters, and so on. Therefore Achilles can never catch the tortoise! Use geometric series to resolve the paradox. At what point does Achilles pass the tortoise?

3. The Koch snowflake is obtained by taking the limit of the sequence of figures below:



   (a) Find the area of the Koch snowflake. Call the area of the triangle in the first figure A.

   (b) Find the perimeter of the Koch snowflake. Call the length of the line segments in the first figure L.

4. Two trains begin 200 miles apart, and are both traveling at 50mph toward each other. A fly begins at one train and travels back and forth between the two trains at 75mph until the trains meet and the fly is crushed. What is the total distance that the fly travels? Here are two ways to answer the question.[1]

   (a) The fly begins 200 miles from the second train. How far does it travel before meeting the second train? From there, how far does it travel before meeting the first train? From there, how far does it travel before meeting the second train again? You should have a geometric series. Find its sum.

   (b) How long before the trains collide? The fly is going 75mph, so how far does it travel?

[1]There is amusing anecdote about this problem and the mathematician John von Neumann. The story is that at a party someone proposed the two trains and a fly question to von Neumann, who immediately gave the correct answer. Thinking that von Neumann had done it the easy way (b), he said "Interesting. Most people try to sum the infinite series." "What do you mean?" von Neumann replied. "That's how I did it."