College of the Holy Cross, Fall 2009
Math 351, Midterm 2
Monday, November 16

1. [15 points] List all cosets of $H = \langle 18 \rangle$ in $\mathbb{Z}_{27}$.

   Note that $H = \{0, 9, 18\}$, and thus there are $27/3 = 9$ distinct cosets of H in $\mathbb{Z}_{27}$. They are $H, 1 + H, 2 + H, \ldots, 8 + H$.

2. [15 points] If G is a group with fewer than 300 elements and G has subgroups of orders 15, 10, and 25, what is the order of G?

   By Lagrange's Theorem, the order of subgroup divides the order of the full group. Thus $|G|$ must be divisible by $10, 15$, and 25. The least common multiple of these three numbers is 150 (it is the first multiple of 25 that is divisible by both 10 and 15), and this is the only number less than 300 that is a multiple of all three. Thus $|G| = 150$.

3. [15 points] Let G_1 and G_2 be groups, H a normal subgroup of G_1, and K a normal subgroup of G_2. Show that $H \oplus K$ is a normal subgroup of $G_1 \oplus G_2$.

   By Lagrange's Theorem, the order of subgroup divides the order of the full group. Thus $|G|$ must be divisible by $10, 15$, and 25. The least common multiple of these three numbers is 150 (it is the first multiple of 25 that is divisible by both 10 and 15), and this is the only number less than 300 that is a multiple of all three. Thus $|G| = 150$.

4. [20 points] Determine the number of elements of order 4 in $D_4 \oplus \mathbb{Z}_{200}$.

We want all elements $(a, b) \in D_4 \oplus \mathbb{Z}_{200}$ with the least common multiple of $|a|$ and $|b|$ equal to 4. There are three possibilities: i) $|a| = 4$ and $|b| = 1, 2$, or 4, ii) $|a| = 2$ and $|b| = 4$, and iii) $|a| = 1$ and $|b| = 4$.

Next, note that in D_4, the elements R_{90} and R_{270} both have order 4, while R_{180} and the four reflections H, V, D, D' all have order 2. The group $\mathbb{Z}_{200}$ is cyclic, and since $1, 2$, and 4 are all divisors of 200, there are $\phi(1) = 1$ elements of order 1, $\phi(2) = 1$ elements of order 2, and $\phi(4) = 2$ elements of order 4. One can also find these elements explicitly: 0 has order 1, 100 has order 2, and 50 and 150 have order 4.

Returning to our cases: in case i) there are $2 \cdot (1 + 1 + 2) = 8$ possibilities. In case ii) there are $5 \cdot 2 = 10$ possibilities, and in case iii) there are $1 \cdot 2 = 2$ possibilities. The total number of elements of order 4 in $D_4 \oplus \mathbb{Z}_{200}$ is thus 20.

5. [15 points] Let G be a group of order p^n. Prove the center of G cannot have order p^{n-1}.

Suppose that $|Z(G)| = p^{n-1}$. Recall that $Z(G) \lhd G$, so that the quotient group $G/Z(G)$ exists. Then

$$|G/Z(G)| = \frac{|G|}{|Z(G)|} = \frac{p^n}{p^{n-1}} = p.$$

Since all groups of prime order are cyclic, we have that $G/Z(G)$ is cyclic. By the $G/Z(G)$ theorem from class, it follows that G is abelian. However, this implies that all elements of G commute with each other, and thus $Z(G) = G$. Hence $|Z(G)| = p^n$. This contradicts our assumption that $|Z(G)| = p^{n-1}$. The contradiction proves that $|Z(G)|$ cannot equal p^{n-1}.

6. [20 points] Suppose that $|G| = 120$, and that H is a normal subgroup of G of order 6. Suppose that $g \in G$ is an element of order 30. What are the possible orders of $gH \in G/H$? Now suppose that K is a subgroup of G of order 3. Prove that $K \subset H$.

First note that $|G/H| = 120/6 = 20$. Since $gH \in G/H$, it follows from Lagrange's theorem that $|gH|$ divides 20. On the other hand, we have

$$(gH)^{30} = g^{30}H = eH = H$$

since $|g| = 30$. Hence $|gH|$ divides 30. This means that gH is one of $1, 2, 5, or 10$ since these are the numbers dividing both 20 and 30. However if $|gH| = 1$ then $g \in H$, but this is impossible since $|g| = 30$ and $|H| = 6$ and the order of a group element must divide the order of the group. If $|gH| = 2$, then $g^2H = H$, so $g^2 \in H$. Thus g^2 has order $1, 2, 3$, or 6, and in all cases $g^{12} = e$. This contradicts $|g| = 30$. Thus the possible orders of gH are 5 and 10.

For the second part of the problem, let $k \in K$. We wish to show that $k \in H$. To do this, it is enough to show $kH = H$, or in other words, $|kH| = 1$ in the group G/H. We know from the first part that $|G/H$ divides 20, so by Lagrange we have that $|kH|$ divides 20. However, since $|K| = 3$, we have $|k|$ divides 3. Thus $(kH)^3 = k^3H = H$, so $|kH|$ divides 3. Since 20 and 3 are relatively prime, the only possibility is $|kH| = 1$, and we are done.