College of the Holy Cross, Spring 2009
Math 244, Practice Midterm 3 solutions
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1. Consider the linear transformation $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ defined by $T(p) = p' + 3p$.

(a) For $\alpha = \{1, x, x^2, x^3\}$ basis for $P_3(\mathbb{R})$ and $\beta = \{1, x, x^2, x^3, x^4\}$ basis for $P_4(\mathbb{R})$, find the matrix of T with respect to α and β, *i.e.*, find $[T]_\alpha^\beta$.

Since $T(1) = 0 + 3 = 3(1) + 0(x) + 0(x^2) + 0(x^3) + 0(x^4)$, the first column of $[T]_\alpha^\beta$ is $(3, 0, 0, 0, 0)$. The full matrix is

$$[T]_\alpha^\beta = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) What is the dimension of Ker(T)? Find a basis for Ker(T).

The echelon form of $[T]_\alpha^\beta$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The associated system of homogeneous equations thus has only the trivial solution, so the dimension of Ker(T) is 0 and a basis for Ker(T) is $\{\mathbf{0}\}$.

(c) What is the dimension of Im(T)? Find a basis for Im(T)

The dimension of Im(T) is 4, since the four columns of $[T]_\alpha^\beta$ are linearly independent. A basis for Im(T) is $\{1, x, x^2, x^3\}$.

(d) Is T injective? Explain your answer.

Yes, since Ker$(T) = \{\mathbf{0}\}$, and we proved in class that this is equivalent to T being injective.

(e) Is T surjective? Explain your answer.

No, since Im(T) has dimension 4, but $\mathbb{R}^5$ has dimension 5.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be injective. Show that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent if and only if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

   Suppose first that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent. To show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, let's suppose that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$ and show that $a_1 = a_2 = a_3 = 0$. Apply T to both sides of $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$ and use that T is linear and thus $T(\mathbf{0}) = \mathbf{0}$. This gives

$$a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + a_3T(\mathbf{v}_3) = \mathbf{0}.$$

   But by hypothesis $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent, and so $a_1 = a_2 = a_3 = 0$.

   Now suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. To show $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent, let's suppose that $a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + a_3T(\mathbf{v}_3) = \mathbf{0}$ and show $a_1 = a_2 = a_3 = 0$. Since $a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + a_3T(\mathbf{v}_3) = \mathbf{0}$ and T is linear, we have

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) = \mathbf{0}.$$

   Since T is injective, we have $\text{Ker}(T) = \{\mathbf{0}\}$ and so $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$. But since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, this implies that $a_1 = a_2 = a_3 = 0$. QED

3. Does there exist a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that $\dim(\text{Im}(T)) = 2\dim(\text{Ker}(T))$? If your answer is yes, give one such transformation. If your answer is not, explain why not.

   No. Suppose $\dim(\text{Im}(T)) = 2\dim(\text{Ker}(T))$. By the dimension theorem, we must have $\dim(\text{Im}(T)) + \dim(\text{Ker}(T)) = 5$. It follows that

$$2\dim(\text{Ker}(T)) + \dim(\text{Ker}(T)) = 5,$$

   and thus

$$3\dim(\text{Ker}(T)) = 5.$$

   But this is impossible, since $\dim(\text{Ker}(T))$ must be an integer.

4. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix with respect with the standard basis of $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$.

   (a) Is T invertible? If so, what is the matrix of T^{-1} with respect to the standard basis?

   Yes. The matrix of T^{-1} with respect to the standard basis is the inverse of $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$, which is

$$\begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & -6 \\ 1 & -1 & 4 \end{bmatrix}.$$

(b) What is the matrix of T with respect to the basis $\alpha = \{(1,1,0),(1,0,1),(0,0,1)\}$ of $\mathbb{R}^3$?

By one of our results from section 2.7, the matrix of T with respect to this basis is $Q^{-1}AQ$, where Q is the matrix of the identity transformation with respect to α and the standard basis. To find the columns of Q, we take the vectors in α and write them in standard basis coordinates. This gives

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

One then computes that

$$Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Now we have that the matrix we want is

$$Q^{-1}AQ = \begin{bmatrix} -1 & -8 & -6 \\ 4 & 13 & 9 \\ -4 & -8 & 5 \end{bmatrix}$$

5. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be given by reflection about the y-axis (that is, the span of $\{(0,1)\}$), and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have the matrix

$$A = \begin{bmatrix} -2 & -1 \\ 2 & 5 \end{bmatrix}$$

with respect to the standard basis of $\mathbb{R}^2$. Find the matrix for TS with respect to the standard basis of $\mathbb{R}^2$. [Hint: calculate the action of TS on each element of the standard basis for $\mathbb{R}^2$.]

First let's find $TS(1,0)$. Since $(1,0)$ is on the x-axis, its reflection about the y-axis is just its negative, so $S(1,0) = (-1,0)$. Now we can find $T(-1,0)$ as follows:

$$\begin{bmatrix} -2 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Thus $TS(1,0) = (2,-2)$. Similarly we find $TS(0,1) = (-1,5)$. Thus the matrix for TS with respect to the standard basis of $\mathbb{R}^2$ is

$$\begin{bmatrix} 2 & -1 \\ -2 & 5 \end{bmatrix}$$

6. Let $S : U \to V$ and $T : V \to W$ be isomorphisms. Prove that TS is injective.

We need to show that $\text{Ker}(TS) = \{\mathbf{0}\}$. Let $\mathbf{x} \in \text{Ker}(TS)$. Thus $T(S(\mathbf{x})) = \mathbf{0}$. Hence $S(\mathbf{x}) \in \text{Ker}(T)$, but since T is an isomorphism we have $\text{Ker}(T) = \{\mathbf{0}\}$, and so $S(\mathbf{x}) = \mathbf{0}$. Now since S is an isomorphism we have $\text{Ker}(S) = \{\mathbf{0}\}$, and so $\mathbf{x} = \mathbf{0}$. This shows that $\text{Ker}(TS) = \{\mathbf{0}\}$. QED

7. For what values of a is the matrix $\begin{bmatrix} 1 & -1 & 1 \\ -a & 0 & -a \\ 1 & a & 0 \end{bmatrix}$ invertible?

Expanding along the second row is a good idea of this matrix, we get that its determinant is $-a^2 + a^2 + a$, which is just a. So as long as $a \neq 0$ the matrix will be invertible.