College of the Holy Cross, Spring 2009
Math 244, Practice Midterm 3
Prof. Jones

1. Consider the linear transformation $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ defined by $T(p) = p' + 3p$.
   (a) For $\alpha = \{1, x, x^2, x^3\}$ basis for $P_3(\mathbb{R})$ and $\beta = \{1, x, x^2, x^3, x^4\}$ basis for $P_4(\mathbb{R})$, find the matrix of T with respect to α and β, *i.e.*, find $[T]_\alpha^\beta$.
   (b) What is the dimension of $\mathrm{Ker}(T)$? Find a basis for $\mathrm{Ker}(T)$.
   (c) What is the dimension of $\mathrm{Im}(T)$? Find a basis for $\mathrm{Im}(T)$
   (d) Is T injective? Explain your answer.
   (e) Is T surjective? Explain your answer.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be injective. Show that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent if and only if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

3. Does there exist a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that $\dim(\mathrm{Im}(T)) = 2\dim(\mathrm{Ker}(T))$? If your answer is yes, give one such transformation. If your answer is not, explain why not.

4. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix with respect with the standard basis of $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$.
   (a) Is T invertible? If so, what is the matrix of T^{-1} with respect to the standard basis?
   (b) What is the matrix of T with respect to the basis $\alpha = \{(1,1,0), (1,0,1), (0,0,1)\}$ of $\mathbb{R}^3$?

5. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be given by reflection about the y-axis (that is, the span of $\{(0,1)\}$), and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have the matrix
$$A = \begin{bmatrix} -2 & -1 \\ 2 & 5 \end{bmatrix}$$
with respect to the standard basis of $\mathbb{R}^2$. Find the matrix for TS with respect to the standard basis of $\mathbb{R}^2$. [Hint: calculate the action of TS on each element of the standard basis for $\mathbb{R}^2$.]

6. Let $S : U \to V$ and $T : V \to W$ be isomorphisms. Prove that TS is injective.

7. For what values of a is the matrix $\begin{bmatrix} 1 & 0 & 1 \\ -a & 0 & -a \\ 1 & a & 0 \end{bmatrix}$ invertible?