

College of the Holy Cross, Spring 2009
Math 244, Practice Midterm 2
Prof. Jones

1. Find a basis for the following subspace of \mathbb{R}^5 .

$$W = \left\{ (x_1, x_2, x_3, x_4, x_5) \mid \begin{array}{rcccc} x_1 & +2x_3 & & +x_5 & = 0 \\ -3x_1 & & +x_4 & & = 0 \\ -x_1 & +4x_3 & +x_4 & +2x_5 & = 0 \end{array} \right\}.$$

This system of equations has echelon form:

$$\begin{array}{rcccc} x_1 & & -\frac{1}{3}x_4 & & = 0 \\ & x_3 & +\frac{1}{6}x_4 & +\frac{1}{2}x_5 & = 0 \\ & & & 0 & = 0 \end{array}$$

Thus the free variables are x_2, x_4 , and x_5 , and we set $x_2 = t_1$, $x_4 = t_2$, and $x_5 = t_3$. The set of solutions is then

$$\left\{ \left(\frac{1}{3}t_2, t_1, -\frac{1}{6}t_2 - \frac{1}{2}t_3, t_2, t_3 \right) \mid t_1, t_2, t_3 \in \mathbb{R} \right\}.$$

Setting $t_1 = 1, t_2 = t_3 = 0$, $t_2 = 1, t_1 = t_3 = 0$, and $t_3 = 1, t_1 = t_2 = 0$ now gives the basis $\{(0, 1, 0, 0, 0), (1/3, 0, -1/6, 1, 0), (0, 0, -1/2, 0, 1)\}$.

2. What is the dimension of the subspace W in question 1?

Three.

3. Let V be a finite-dimensional vector space, and let S be a linearly independent subset of V . Let S' be a proper subset of S (this means that $S' \subseteq S$ and $S' \neq S$). Prove that S' cannot be a basis for V .

By Theorem 1.6.6, S can be extended to a basis B for V . Thus the number of elements in B is at least as large as the number of elements in S . However, the number of elements in S' is strictly smaller than the number of elements in S , and thus strictly smaller than the number of elements in B . Since every basis contains the same number of elements, S' cannot be a basis.

4. (a) Find a basis for the following subspace of $P_4(\mathbb{R})$.

$$W = \{p \in P_4(\mathbb{R}) \mid p(1) = p(-1) = 0\}.$$

Note that W is the same as

$$\{a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in P_4(\mathbb{R}) \mid a_4 + a_3 + a_2 + a_1 + a_0 = 0, a_4 - a_3 + a_2 - a_1 + a_0 = 0\}.$$

We thus need to put the system of equations

$$\begin{array}{cccccc} a_4 & +a_3 & +a_2 & +a_1 & +a_0 & = 0 \\ a_4 & -a_3 & +a_2 & -a_1 & +a_0 & = 0 \end{array}$$

in echelon form. We get

$$\begin{array}{cccccc} a_4 & +a_3 & +a_2 & +a_1 & +a_0 & = 0 \\ & -2a_3 & & -2a_1 & & = 0 \end{array}$$

and this gives the echelon form

$$\begin{array}{cccccc} a_4 & & +a_2 & & +a_0 & = 0 \\ & a_3 & & +a_1 & & = 0 \end{array}$$

Thus our free variables are a_2 , a_1 , and a_0 . Set $a_0 = t_1$, $a_1 = t_2$, and $a_2 = t_3$. Using the ordering $(a_0, a_1, a_2, a_3, a_4)$, the set of solutions is then

$$\{(t_1, t_2, t_3, -t_2, -t_3 - t_1) \mid t_1, t_2 \in \mathbb{R}\}.$$

Setting $t_1 = 1, t_2 = t_3 = 0$, $t_2 = 1, t_1 = t_3 = 0$, and $t_3 = 1, t_1 = t_2 = 0$ now gives the vectors $\{(1, 0, 0, 0, -1), (0, 1, 0, -1, 0), (0, 0, 1, 0, -1)\}$. This corresponds to the basis $\{-x^4 + 1, -x^3 + x, -x^4 + x^2\}$.

(b) What is the dimension of W ?

Three.

(c) Extend the basis you found in part (a) to a basis of $P_4(\mathbb{R})$.

We use the method of Theorem 1.6.6. The set $\{1, x, x^2, x^3, x^4\}$ spans $P_4(\mathbb{R})$ (in fact, it is a basis of $P_4(\mathbb{R})$). If add the element 1 to the basis found in part (a), the resulting set is linearly independent, as you can check by taking $c_1(-x^4 + 1) + c_2(-x^3 + x) + c_3(-x^4 + x^2) + c_4(1) = 0$ and reducing the resulting matrix to echelon form. Adding the element x to this new set again preserves linear independence, and so the set

$$\{-x^4 + 1, -x^3 + x, -x^4 + x^2, 1, x\}$$

is a basis for $P_4(\mathbb{R})$, since it consists of five linearly independent elements and the dimension of $P_4(\mathbb{R})$ is 5.

5. Determine whether the following mappings are linear transformations. Either prove that a given map is linear or give a counterexample to show it's not linear.

(a) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T((x_1, x_2)) = (2x_1, x_1 + 4, 5x_2)$

Not linear since $T((0, 0)) = (0, 4, 0) \neq (0, 0, 0)$.

(b) $T : P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ defined by $T(a_2x^2 + a_1x + a_0) = a_0x^3 + (a_1 - a_0)x^2 + 3a_2 - (1/2)a_0$

Linear. Let $\vec{u} = u_2x^2 + u_1x + u_0$ and $\vec{v} = v_2x^2 + v_1x + v_0$. Then

$$\begin{aligned} T(a\vec{u} + b\vec{v}) &= (au_0 + bv_0)x^3 + (au_1 + bv_1 - au_0 - bv_0)x^2 + 3(au_2 + bv_2) - (1/2)(au_0 + bv_0) \\ &= a(u_0x^3 + (u_1 - u_0)x^2 + 3u_2 - (1/2)u_0) + b(v_0x^3 + (v_1 - v_0)x^2 + 3v_2 - (1/2)v_0) \\ &= aT(\vec{u}) + bT(\vec{v}) \end{aligned}$$

6. (a) Consider the mapping $T : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ defined by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & c - b \\ b + 2d - 3c & d + 4a \end{pmatrix}.$$

Prove that T is a linear transformation.

Let

$$\vec{u} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

Then

$$\begin{aligned} T(a\vec{u} + b\vec{v}) &= \begin{pmatrix} au_{11} + bv_{11} + au_{12} + bv_{12} & au_{21} + bv_{21} - (au_{12} + bv_{12}) \\ au_{12} + bv_{12} + 2(au_{22} + bv_{22}) - 3(au_{21} + bv_{21}) & au_{22} + bv_{22} + 4(au_{11} + bv_{11}) \end{pmatrix} \\ &= a \begin{pmatrix} u_{11} + u_{12} & u_{21} - u_{12} \\ u_{12} + 2u_{22} - 3u_{21} & u_{22} + 4u_{11} \end{pmatrix} + b \begin{pmatrix} v_{11} + v_{12} & v_{21} - v_{12} \\ v_{12} + 2v_{22} - 3v_{21} & v_{22} + 4v_{11} \end{pmatrix} \\ &= aT(\vec{u}) + bT(\vec{v}) \end{aligned}$$

(b) Given the basis $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $M_{2 \times 2}(\mathbb{R})$,

give the matrix $[T]_{\alpha}^{\alpha}$ of T with respect to the basis α .

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus the first column of $[T]_{\alpha}^{\alpha}$ is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

Proceeding similarly with the other columns, we get

$$[T]_{\alpha}^{\alpha} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

7. The mapping $T : \mathbb{R}^2 \rightarrow P_2(\mathbb{R})$ given by $T((a_1, a_2)) = (a_1 + a_2)x^2 + a_2x + a_1$ is a linear transformation. Prove that $\alpha = \{(1, 2), (-1, 0)\}$ is a basis for \mathbb{R}^2 and $\beta = \{x^2 + 2, x^2 + x, 1\}$ is a basis for $P_2(\mathbb{R})$. Then find the matrix $[T]_{\alpha}^{\beta}$

First note that α is a basis for \mathbb{R}^2 since α is linearly independent (it is a set of two vectors and neither is a multiple of the other) and the dimension of \mathbb{R}^2 is two. Similarly, one can check with a matrix computation that β is linearly independent, and since $P_2(\mathbb{R})$ has dimension 3 this shows that β is a basis.

To compute the first column of $[T]_{\alpha}^{\beta}$, we find $T((1, 2))$ and write it in β -coordinates. We have

$$T((1, 2)) = 3x^2 + 2x + 1 = 1(x^2 + 2) + 2(x^2 + x) - 1(1)$$

and thus the first column of $[T]_{\alpha}^{\beta}$ is

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Note that to find the β -coordinates of $3x^2 + 2x + 1$, you can either eyeball a solution as follows: only $x^2 + x$ involves x , and so we must have $2(x^2 + x)$, and then the only other term with x^2 is $x^2 + 2$, so we must add on $1(x^2 + 2)$ to get $3x^2$, then we have a constant term of 2, so we must subtract $1(1)$. Or you can take the more systematic approach of writing

$$3x^2 + 2x + 1 = a_1(x^2 + 2) + a_2(x^2 + x) + a_3(1),$$

which leads to the system of equations

$$\begin{array}{rcl} a_2 & +a_1 & = 3 \\ a_2 & & = 2 \\ a_3 & +2a_1 & = 1 \end{array}$$

which you can solve.

In the same way we can find the second column of $[T]_{\alpha}^{\beta}$, which gives

$$[T]_{\alpha}^{\beta} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$$