College of the Holy Cross, Spring 2009
Math 244, Practice Midterm 2
Prof. Jones

1. Find a basis for the following subspace of $\mathbb{R}^5$.

$$W = \left\{ (x_1, x_2, x_3, x_4, x_5) \;\middle|\; \begin{array}{rrrrr} x_1 & +2x_3 & & +x_5 & = 0 \\ -3x_1 & & +x_4 & & = 0 \\ -x_1 & +4x_3 & +x_4 & +2x_5 & = 0 \end{array} \right\}.$$

2. What is the dimension of the subspace W in question 1?

3. Let V be a finite-dimensional vector space, and let S be a linearly independent subset of V. Let S' be a proper subset of S (this means that $S' \subseteq S$ and $S' \neq S$). Prove that S' cannot be a basis for V.

4. (a) Find a basis for the following subspace of $P_4(\mathbb{R})$.

$$W = \{p \in P_4(\mathbb{R}) \mid p(1) = p(-1) = 0\}.$$

   (b) What is the dimension of W?

   (c) Extend the basis you found in part (a) to a basis of $P_4(\mathbb{R})$.

5. Determine whether the following mappings are linear transformations. Either prove that a given map is linear or give a counterexample to show it's not linear.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T((x_1, x_2)) = (2x_1, x_1 + 4, 5x_2)$

(b) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(a_2x^2 + a_1x + a_0) = a_0x^3 + (a_1 - a_0)x^2 + 3a_2 - (1/2)a_0$

6. (a) Consider the mapping $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & c-b \\ b+2d-3c & d+4a \end{pmatrix}.$$

   Prove that T is a linear transformation.

   (b) Given the basis $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $M_{2\times 2}(\mathbb{R})$,

   give the matrix $[T]_\alpha^\alpha$ of T with respect to the basis α.

7. The mapping $T : \mathbb{R}^2 \to P_2(\mathbb{R})$ given by $T((a_1, a_2)) = (a_1 + a_2)x^2 + a_2x + a_1$ is a linear transformation. Prove that $\alpha = \{(1, 2), (-1, 0)\}$ is a basis for $\mathbb{R}^2$ and $\beta = \{x^2 + 2, x^2 + x, 1\}$ is a basis for $P_2(\mathbb{R})$. Then find the matrix $[T]_\alpha^\beta$