College of the Holy Cross, Spring 2009
Math 244, Practice Midterm 1
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1. For each of the following subsets W of a vector space V, determine if W is a subspace of V. In each case either prove that W is a subspace or give a concrete reason why it is not a subspace.

(b) $V = \mathbb{R}^4$, and $W = \{(x_1, x_2, x_3, x_4) \,|\, x_1 = x_3 - x_2, \text{ and } x_4 = x_1 x_3\}$

It seems unlikely to be a subspace since the second condition $x_4 = x_1x_3$ involves multiplication of variables and so is not linear. Indeed, we can show W is not closed under addition: $(1, 2, 3, 3) \in W$ and $(-1, 2, 1, -1) \in W$ but their sum $(0, 4, 4, 2)$ is not in W since it fails the condition $x_4 = x_1x_3$. You could also proceed by showing that W is not closed under scalar multiplication: $(1, 2, 3, 3) \in W$, but $2(1, 2, 3, 3) = (2, 4, 6, 6)$ is not in W.

(c) $V = M_{2\times 2}(\mathbb{R})$, and $W = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \,|\, a_{21} = 0 \right\}$

This is a subspace. Let $\mathbf{x}$ and $\mathbf{y}$ be in W, so that

$$\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

with $a_{21} = b_{21} = 0$. Then

$$c\mathbf{x} + \mathbf{y} = \begin{bmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} \\ ca_{21} + b_{21} & ca_{22} + b_{22} \end{bmatrix}$$

and the 21 entry of this matrix is $ca_{21} + b_{21}$, which is 0 since $a_{21} = b_{21} = 0$. Thus $c\mathbf{x} + \mathbf{y} \in W$, and we've proven that W is a subspace.

2. Given subspaces W_1 and W_2 of a vector space V, recall that

$$W_1 + W_2 = \{\mathbf{x} \in V \,|\, \mathbf{x} = \mathbf{w_1} + \mathbf{w_2} \text{for some } \mathbf{w_1} \in W_1 \text{ and } \mathbf{w_2} \in W_2\}.$$

Prove that $W_1 + W_2$ is a subspace of V.

Let $\mathbf{x}$ and $\mathbf{y}$ be in $W_1 + W_2$, meaning that $\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$ for some $\mathbf{x_1} \in W_1$, $\mathbf{x_2} \in W_2$ and $\mathbf{y} = \mathbf{y_1} + \mathbf{y_2}$ for some $\mathbf{y_1} \in W_1$, $\mathbf{y_2} \in W_2$. Then

$$c\mathbf{x} + \mathbf{y} = c\mathbf{x_1} + c\mathbf{x_2} + \mathbf{y_1} + \mathbf{y_2} = (c\mathbf{x_1} + \mathbf{y_1}) + (c\mathbf{x_2} + \mathbf{y_2}).$$

Since W_1 is a subspace, $c\mathbf{x_1} + \mathbf{y_1} \in W_1$, and since W_2 is a subspace, $c\mathbf{x_2} + \mathbf{y_2} \in W_2$. We've thus written $c\mathbf{x} + \mathbf{y}$ as an element of W_1 plus an element of W_2, showing that $c\mathbf{x} + \mathbf{y} \in W_1 + W_2$.

3. Let $V = \mathbb{R}^2$ and $S = \{(2,2),(2,-2)\}$.

(a) Show that $(1,2)$ is in Span(S). We need to find weights x_1 and x_2 such that $x_1(2,2) + x_2(2,-2) = (1,2)$. Equating coordinates gives

$$\begin{array}{rrcl} 2x_1 & +2x_2 & = & 1 \\ 2x_1 & -2x_2 & = & 2 \end{array}$$

Adding -1 times the first equation to the second gives

$$\begin{array}{rrcl} 2x_1 & +2x_2 & = & 1 \\ & -4x_2 & = & 1 \end{array}$$

We now divide the first equation by 2 and multiply the second equation by $-1/4$. Finally, we add -1 times the new second equation to the first equation, and this gives the echelon form

$$\begin{array}{rrcr} x_1 & & = & 3/4 \\ & x_2 & = & -1/4 \end{array}$$

We have thus shown that $(1,2) = \frac{3}{4}(2,2) - \frac{1}{4}(2,-2)$, proving that $(1,2)$ is in Span(S).

(b) Show that every vector (b_1, b_2) in $\mathbb{R}^2$ is in Span(S).

The calculation is very similar to that of part (a). Let $(b_1, b_2) \in \mathbb{R}^2$, and look for weights x_1, x_2 satisfying $x_1(2,2) + x_2(2,-2) = (b_1, b_2)$. Equating coordinates gives

$$\begin{array}{rrcl} 2x_1 & +2x_2 & = & b_1 \\ 2x_1 & -2x_2 & = & b_2 \end{array}$$

Adding -1 times the first equation to the second gives

$$\begin{array}{rrcc} 2x_1 & +2x_2 & = & b_1 \\ & -4x_2 & = & b_2 - b_1 \end{array}$$

We now divide the first equation by 2 and multiply the second equation by $-1/4$. Finally, we add -1 times the new second equation to the first equation, and this gives the echelon form

$$\begin{array}{rrcc} x_1 & & = & (1/2)b_1 + 1/4(b_2 - b_1) \\ & x_2 & = & -1/4(b_2 - b_1) \end{array}$$

Thus the system of equations has a solution, proving that (b_1, b_2) is in Span(S). Thus Span(S) $= \mathbb{R}^2$.

4. Let V be a vector space, and S a non-empty subset of V. Assume that each vector in Span(S) can be written in one and only one way as a linear combination of vectors in S. Show that S is linearly independent.

   Suppse that we have $a_1\mathbf{x_1} + \cdots + a_n\mathbf{x_n} = \mathbf{0}$ with $\mathbf{x_i} \in S$. We know that $\mathbf{0}$ can also be written as the linear combination

$$0\mathbf{x_1} + \cdots + 0\mathbf{x_n}.$$

   Since $\mathbf{0} \in \text{Span}(S)$, our hypothesis gives us that there is only one way to write $\mathbf{0}$ as a linear combination of vectors in S. Therefore we must have $a_1 = 0, \ldots, a_n = 0$. Hence S is linearly independent.

5. Determine whether each of the following sets of vectors is linearly independent or linearly dependent. In each case prove your assertion.

   (a) $S = \{1, x, x^2\}$ in $P_2(\mathbb{R})$.
   Linearly independent: if $a_1 \cdot 1 + a_2 \cdot x + a_3 \cdot x^2 = 0$, then by equating coefficients (x^2 term, x term, constant term), we immediately get $a_1 = a_2 = a_3 = 0$.

   (b) $S = \{(0, 2, 4), (0, 1, 3), (1, 1, 4)\}$ in $\mathbb{R}^3$.
   Linearly independent: if $a_1(0, 2, 4) + a_2(0, 1, 3) + a_3(1, 1, 4) = (0, 0, 0)$, then we have

$$\begin{array}{rcrcrcl} & & & & a_3 & = & 0 \\ 2a_1 & + & a_2 & + & a_3 & = & 0 \\ 4a_1 & + & 3a_2 & + & 4a_3 & = & 0 \end{array}$$

   We could put this system into echelon form in order to find all solutions, but in this case there is a slightly shorter way. Adding -2 times equation 2 to equation 3 gives

$$\begin{array}{rcrcrcl} & & & & a_3 & = & 0 \\ 2a_1 & + & a_2 & + & a_3 & = & 0 \\ & & a_2 & + & 2a_3 & = & 0 \end{array}$$

   Since $a_3 = 0$, substitution into the third equation shows that $a_2 = 0$, and then substitution into second equation shows $a_1 = 0$.

   (c) $S = \{e^x, e^{2x}, \sqrt{x}\}$ in $F(\mathbb{R})$.

   Linearly independent: if $a_1 e^x + a_2 e^{2x} + a_3\sqrt{x} = 0$ (as functions), then this equation must hold for all values of x. Plugging in $x = 0$, $x = 1$, and $x = 4$ yields

$$\begin{array}{rcrcrcl} a_1 & + & a_2 & & & = & 0 \\ ea_1 & + & e^2 a_2 & + & a_3 & = & 0 \\ e^4 a_1 & + & e^{16} a_2 & + & 2a_3 & = & 0 \end{array}$$

As in part (b), we could put this in echelon form, but we go for a shorter method. Adding $-e$ times the first equation to the second equation and $-e^4$ times the first equation to the third equation gives

$$\begin{array}{ccccccc} a_1 & + & a_2 & & & = & 0 \\ & & (e^2-e)a_2 & + & a_3 & = & 0 \\ & & (e^{16}-e)a_2 & + & 2a_3 & = & 0 \end{array}$$

Now adding -2 times the second equation to the third equation gives

$$\begin{array}{ccccccc} a_1 & + & a_2 & & & = & 0 \\ & & (e^2-e)a_2 & + & a_3 & = & 0 \\ & & (e^{16}-2e^2+e)a_2 & & & = & 0 \end{array}$$

The third equation gives us $a_2 = 0$, and then substitution into the second and first equations gives $a_3 = 0$ and $a_1 = 0$.

6. Find a parametrization of the set of solutions of the following system of linear equations.

$$\begin{array}{ccccc} x_1 & +2x_2 & & +x_4 & = 0 \\ -3x_1 & & +x_3 & & = 1 \\ -x_1 & +4x_2 & +x_3 & +2x_4 & = 1 \end{array}$$

To find a parameterization for the set of solutions, we need to put this system into echelon form. Adding 3 times the first equation to the second equation and 1 times the first equation to the third equation gives

$$\begin{array}{ccccc} x_1 & +2x_2 & & +x_4 & = 0 \\ & +6x_2 & +x_3 & +3x_4 & = 1 \\ & +6x_2 & +x_3 & +3x_4 & = 1 \end{array}$$

Adding -1 times the second equation to the third equation and -1/3 times the second equation to the first equation and then multiplying the second equation by 1/6 gives

$$\begin{array}{cccccc} x_1 & & -(1/3)x_3 & & = & -1/3 \\ & +x_2 & +(1/6)x_3 & +(1/2)x_4 & = & 1/6 \\ & & & 0 & = & 0 \end{array}$$

This is now in echelon form. We see that the basic variables are x_1 and x_2, while the free variables are x_3 and x_4. Setting $x_3 = t_1$ and $x_4 = t_2$, we get that the set of solutions is

$$\left\{ \left(\frac{1}{3}t_1 - \frac{1}{3}, -\frac{1}{6}t_1 - \frac{1}{2}t_2 + \frac{1}{6}, t_1, t_2 \right) \in \mathbb{R}^4 \mid t_1, t_2 \in \mathbb{R} \right\}$$