

College of the Holy Cross, Spring 2009
Math 244, Practice Final
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1. Let $V = P_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3. Consider the subset $W = \{p \in V \mid p(2) = 0 \text{ and } p'(2) = 0\}$.

(a) Show that W is a subspace of V .

Let $p(x)$ and $q(x)$ be in W , and $a, b \in \mathbb{R}$. This means that $p(2) = q(2) = 0$ and $p'(2) = q'(2) = 0$. Then $(ap + bq)(2) = a(p(2)) + b(q(2)) = 0 + 0 = 0$ and $(ap + bq)'(2) = a(p'(2)) + b(q'(2))$ (since derivatives are additive and $(cf(x))' = cf'(x)$) and this equals 0. So $(ap + bq)(x) \in W$, and this proves that W is a subspace of V .

(b) Find a basis for W . What is the dimension of W ?

Write $p(x) \in V$ as $a_0x^3 + a_1x^2 + a_2x + a_3$. Then we can rewrite W as

$$\{a_0x^3 + a_1x^2 + a_2x + a_3 \mid 8a_0 + 4a_1 + 2a_2 + a_3 = 0 \text{ and } 12a_0 + 4a_1 + a_2 = 0\}.$$

This is a system of equations corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} 8 & 4 & 2 & 1 & 0 \\ 12 & 4 & 1 & 0 & 0 \end{array} \right]$$

In echelon form, this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/4 & -1/4 & 0 \\ 0 & 1 & 1 & 3/4 & 0 \end{array} \right]$$

The free variables are thus a_2 and a_3 . Setting $a_2 = t_1$ and $a_3 = t_2$, we have $a_0 = \frac{1}{4}t_1 + \frac{1}{4}t_2$ and $a_1 = -t_1 - \frac{3}{4}t_2$. In vector form, this is equal to the span of $\{(1/4, -1, 1, 0), (1/4, -3/4, 0, 1)\}$ (setting $t_1 = 1, t_2 = 0$ and then vice versa). Therefore a basis for W is

$$\{1/4x^3 - x^2 + x, 1/4x^3 - 3/4x^2 + 1\}.$$

- (c) Is the subset $W = \{p \in V \mid p(2) = 0 \text{ and } p'(2) = 1\}$ a subspace of V ? Either prove it is or explain why it is not.

No, since W does not contain the zero polynomial (which has derivative equal to 0 and thus cannot satisfy $p'(2) = 1$).

2. (a) Explain what it means for a set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ to span a vector space V .

Every vector $v \in V$ can be written as some linear combination of $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. In other words, each $v \in V$ can be written $a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$ for some $a_1, \dots, a_n \in \mathbb{R}$.

- (b) Show that that $\{1 + x + x^3, x^2 + 2x^3, 6 + x^2 + 2x^3, 4 + x\}$ spans $P_3(\mathbb{R})$.

Let $b_0x^3 + b_1x^2 + b_2x + b_3$ be an arbitrary vector in $P_3(\mathbb{R})$. We need to find a_0, a_1, a_2, a_3 such that

$$a_0(1 + x + x^3) + a_1(x^2 + 2x^3) + a_2(6 + x^2 + 2x^3) + a_3(4 + x) = b_0x^3 + b_1x^2 + b_2x + b_3.$$

From the above, we get one equation for each power of x ; for instance, the constant terms give $a_0 + 6a_2 + 4a_3 = b_3$. Thinking of the a_i as variables, this gives us a system of equations with this augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 6 & 4 & b_3 \\ 1 & 0 & 0 & 1 & b_2 \\ 0 & 1 & 1 & 0 & b_1 \\ 1 & 2 & 2 & 0 & b_0 \end{array} \right]$$

In echelon form this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array} \right]$$

where the $*$ denote expressions involving the b_i (it does not matter what they are). Since this system has no inconsistent equations (those of the form $0 = c$ for some $c \neq 0$), it must have a solution. Thus there exist choices for a_0, a_1, a_2, a_3 that work, meaning that $b_0x^3 + b_1x^2 + b_2x + b_3$ is in the span of $\{1 + x + x^3, x^2 + 2x^3, 6 + x^2 + 2x^3, 4 + x\}$. This shows that $\{1 + x + x^3, x^2 + 2x^3, 6 + x^2 + 2x^3, 4 + x\}$ spans $P_3(\mathbb{R})$.

3. Determine whether the following sets are linearly independent. If so, prove they are, and if not, explain why not.

(a) $\{(1, 2, 1, 0), (3, 1, 0, 0), (0, 0, -1, -2), (-1, 3, 3, 2)\} \subset \mathbb{R}^4$

Look for solutions to the system of equations

$$a_0(1, 2, 1, 0) + a_1(3, 1, 0, 0) + a_2(0, 0, -1, -2) + a_3(-1, 3, 3, 2) = (0, 0, 0, 0).$$

This give the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & 0 \\ 2 & 1 & 0 & 3 & 0 \\ 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{array} \right]$$

In echelon form this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Since there is a free variable (a_3), there will be non-zero solutions, so the set of vectors is linearly dependent. Indeed, taking $a_3 = 1$ (and so $a_0 = -2$, $a_1 = 1$, and $a_2 = 1$) gives the dependence

$$-2(1, 2, 1, 0) + (3, 1, 0, 0) + (0, 0, -1, -2) + (-1, 3, 3, 2) = (0, 0, 0, 0).$$

(b) $\{1 + x + x^3, x^2 + 2x^3, 6 + x^2 + 2x^3, 4 + x\} \subset P_3(\mathbb{R})$

Here are two ways to do this: First, you could note that from 2(b) this set spans $P_3(\mathbb{R})$. If the set is not linearly independent, then one vector is a linear combination of the others, and we can remove it to get a set that still spans $P_3(\mathbb{R})$. Keep doing this until you get a linearly independent set; this set will be a basis of $P_3(\mathbb{R})$. But the dimension of $P_3(\mathbb{R})$ is 4, so every basis has 4 element. Thus it must be that the full set above is linearly independent.

Second, using the same technique as in part (a), but this time getting one equation for each power of x (rather than one equation for each coordinate as in part (a)), we get a system of equations corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 6 & 4 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \end{array} \right]$$

In echelon form this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Thus the only solution is $a_0 = a_1 = a_2 = a_3 = 0$, whence the set is linearly independent.

4. Let S be a linearly independent subset of a vector space V . Show that every subset of S is also linearly independent.

Let $S' \subseteq S$, and suppose

$$a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n = \mathbf{0}$$

with $\mathbf{v}_i \in S'$ and $a_i \in \mathbb{R}$ for all i . Since $S' \subseteq S$, this is also a linear dependence among elements of S , and since S is linearly independent, we have $a_i = 0$ for all i . Therefore S' is linearly independent.

5. Define the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(\mathbf{e}_1) = \mathbf{e}_1 - \mathbf{e}_2$, $T(\mathbf{e}_2) = \mathbf{e}_2 - \mathbf{e}_3$ and $T(\mathbf{e}_3) = \mathbf{e}_3 - \mathbf{e}_1$.

- (a) What is the matrix of T with respect to the standard basis?

Since $T(1, 0, 0) = (1, -1, 0)$, $T(0, 1, 0) = (0, 1, -1)$, $T(0, 0, 1) = (-1, 0, 1)$, the matrix for T with respect to the standard basis is

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

- (b) What is $T(x_1, x_2, x_3)$ for an arbitrary vector $\mathbf{x} = (x_1, x_2, x_3)$?

$$T(x_1, x_2, x_3) = x_1T(1, 0, 0) + x_2T(0, 1, 0) + x_3T(0, 0, 1) = x_1(1, -1, 0) + x_2(0, 1, -1) + x_3(-1, 0, 1),$$

and this is equal to $(x_1 - x_3, -x_1 + x_2, -x_2 + x_3)$.

- (c) Find a basis for the kernel of T . What is the dimension of $\text{Ker}(T)$? To find a basis for the kernel of T , we look for solutions to $T(x_1, x_2, x_3) = (0, 0, 0)$. This leads to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right]$$

In echelon form this matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

There is one free variable, so the dimension of $\text{Ker}(T)$ is 1. The solutions are the span of $\{(1, 1, 1)\}$, and so a basis for $\text{Ker}(T)$ is $\{(1, 1, 1)\}$.

- (d) Find a basis for the image of T . What is the dimension of $\text{Im}(T)$?

A basis for the image of T consists of the columns of the matrix for T corresponding to basic variables in the echelon form. In this case, the leading entries of the echelon form are in the first two columns, and thus $\{(1, -1, 0), (0, 1, -1)\}$ is a basis for the image of T . The dimension of $\text{Im}(T)$ is two.

- (e) What is the matrix of T with respect to the basis $\alpha = \{(1, 1, 1), (1, 0, 2), (-1, -1, 4)\}$ in the domain and the basis $\beta = \{(1, -1, 0), (2, 0, 1), (0, 0, 1)\}$ in the target.

From part (b), we have $T(1, 1, 1) = (0, 0, 0)$, $T(1, 0, 2) = (-1, -1, -2)$, and $T(-1, -1, 4) = (-5, 0, 5)$. To find the matrix requested, we need to write each of these vectors in their β -coordinates. In this case we can eyeball them:

$$\begin{aligned}(0, 0, 0) &= 0(1, -1, 0) + 0(2, 0, 1) + 0(0, 0, 1) \\ (-1, -1, -2) &= 1(1, -1, 0) + (-1)(2, 0, 1) + 1(0, 0, 1) \\ (-5, 0, 5) &= 0(1, -1, 0) + (-5/2)(2, 0, 1) + 15/2(0, 0, 1)\end{aligned}$$

Note that to find these systematically, you can set up a system of equations. For instance, to find the β -coordinates of $(-1, -1, -2)$, you need to solve the system corresponding to

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{array} \right]$$

We now have

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -5/2 \\ 0 & 1 & 15/2 \end{bmatrix}$$

6. Define a mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}$ by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{a} \rangle = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ (for $\mathbf{x} = (x_1, x_2, \dots, x_n)$). Show that T is a linear transformation. (Thus, the inner product is linear in the first component.)

To show that T is linear, we must show that $T(b\mathbf{x} + c\mathbf{y}) = bT(\mathbf{x}) + cT(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $b, c \in \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$

$$\begin{aligned}T(b\mathbf{x} + c\mathbf{y}) &= \langle b\mathbf{x} + c\mathbf{y}, \mathbf{a} \rangle \\ &= a_1(bx_1 + cy_1) + \cdots + a_n(bx_n + cy_n) \\ &= b(a_1x_1 + \cdots + a_nx_n) + c(a_1y_1 + \cdots + a_ny_n) \\ &= b\langle \mathbf{x}, \mathbf{a} \rangle + c\langle \mathbf{y}, \mathbf{a} \rangle \\ &= bT(\mathbf{x}) + cT(\mathbf{y})\end{aligned}$$

7. Let $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $S(a_1, a_2) = (2a_1 + 3a_2, -a_1 + a_2)$ and $T : \mathbb{R}^2 \rightarrow P_2(\mathbb{R})$ be given by $T(a_1, a_2) = a_1x^2 - 3a_2x + 5a_1 - 3a_2$. Let α be the standard basis for \mathbb{R}^2 and $\beta = \{1, x, x^2\}$. Find $[TS]_{\alpha}^{\beta}$.

One method is to note that

$$[S]_{\alpha}^{\alpha} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

and

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \\ 5 & -3 \end{bmatrix}$$

And thus

$$[TS]_{\alpha}^{\beta} = [T]_{\alpha}^{\beta}[S]_{\alpha}^{\alpha} = \begin{bmatrix} 2 & 3 \\ 3 & -3 \\ 13 & 12 \end{bmatrix}$$

8. Let $T : \mathbb{R}^7 \rightarrow \mathbb{R}^4$ be a linear transformation. Without using the dimension theorem, prove that $\dim(\ker(T)) \geq 3$. (Hint: consider the matrix of T with respect to the standard bases of \mathbb{R}^7 and \mathbb{R}^4 .)

Let A be the matrix of T with respect to the standard bases of \mathbb{R}^7 and \mathbb{R}^4 . Then finding the kernel of T is the same as finding the solutions to the matrix equation $A\mathbf{x} = \mathbf{0}$. But since A has 7 rows and 4 columns, there must be at least three free variables in this system of equations, showing that $\ker(T)$ has dimension at least 3.

9. Consider the linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ whose matrix with respect with the standard basis is

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 3 & 0 & -2 \\ 0 & 0 & 3 & 0 \\ 6 & 0 & 0 & -1 \end{bmatrix}.$$

- (a) Find the eigenvalues for T .

The characteristic polynomial is $(3 - \lambda)^3(-1 - \lambda)$, so the eigenvalues are 3 (with multiplicity 3) and -1.

- (b) For each eigenvalue, find a basis for the corresponding eigenspace.

For $\lambda = 3$, we need to find a basis for all solutions of the system of equations corresponding to

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & -4 & 0 \end{array} \right]$$

In echelon form this is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

And a basis for the set of solutions is $\{(0, 1, 0, 0), (0, 0, 1, 0)\}$.

For $\lambda = -1$, we need to find a basis for all solutions of the system of equations corresponding to

$$\left[\begin{array}{cccc|c} 4 & 0 & 0 & 0 & 0 \\ 6 & 4 & 0 & -2 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 \end{array} \right]$$

In echelon form this is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

And a basis for the set of solutions is $\{(0, 1/2, 0, 1)\}$.

- (c) Is T diagonalizable? If yes, give a basis of \mathbb{R}^3 consisting of eigenvectors for T . If not, explain why not.

No, since the eigenspace associated to $\lambda = 3$ has dimension two, but the multiplicity of this eigenvalue is 3.

10. Let V be a vector space and let $T : V \rightarrow V$ be a linear transformation with the property that $T^2 = I$, i.e. $T \circ T$ is the identity transformation.

- (a) Show that if λ is an eigenvalue of T , then $\lambda = 1$ or $\lambda = -1$.

If λ is an eigenvalue of T , then for some non-zero \mathbf{x} , we have $T(\mathbf{x}) = \lambda\mathbf{x}$. Applying T again to both sides gives $T(T(\mathbf{x})) = T(\lambda\mathbf{x})$. But $T(T(\mathbf{x})) = \mathbf{x}$ by hypothesis, and $T(\lambda\mathbf{x}) = \lambda T(\mathbf{x}) = \lambda^2\mathbf{x}$. Putting this all together gives

$$\mathbf{x} = \lambda^2\mathbf{x}.$$

Since $\mathbf{x} \neq \mathbf{0}$, this means that $\lambda^2 = 1$, and thus $\lambda = \pm 1$.

- (b) Show that the eigenspaces satisfy $E_1 \cap E_{-1} = \{\mathbf{0}\}$.

Suppose that $\mathbf{x} \in E_1$ and $\mathbf{x} \in E_{-1}$. Then $T(\mathbf{x}) = \mathbf{x}$ and $T(\mathbf{x}) = -\mathbf{x}$. Thus $\mathbf{x} = -\mathbf{x}$, and so $2\mathbf{x} = \mathbf{0}$, implying that $\mathbf{x} = \mathbf{0}$. This shows that $E_1 \cap E_{-1} = \{\mathbf{0}\}$.

- (c) Assume that $V = E_1 + E_{-1}$ (i.e. every vector in V can be written as the sum of a vector in E_1 and E_{-1}). Does this mean that T must be diagonalizable? Explain.

Yes. Let $\{x_1, \dots, x_n\}$ be a basis for E_1 and let $\{y_1, \dots, y_n\}$ be a basis for E_{-1} . By a theorem from section 8.2, the union of these two sets, $\{x_1, \dots, x_n, y_1, \dots, y_n\}$, remains linearly independent. But since $V = E_1 + E_{-1}$, every vector in V can be written as a linear combination of $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ (why is this?). Thus $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ is a basis for V consisting of eigenvectors for T , and hence T is diagonalizable.