College of the Holy Cross, Spring 2009
Math 244, Practice Final
Prof. Jones

1. Let $V = P_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3. Consider the subset $W = \{p \in V \,|\, p(2) = 0 \text{ and } p'(2) = 0\}$.

   (a) Show that W is a subspace of V.

   (b) Find a basis for W. What is the dimension of W?

   (c) Is the subset $W = \{p \in V \,|\, p(2) = 0 \text{ and } p'(2) = 1\}$ a subspace of V? Either prove it is or explain why it is not.

2. (a) Explain what it means for a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to span a vector space V.

   (b) Show that that $\{1 + x + x^3, x^2 + 2x^3, 6 + x^2 + 2x^3, 4 + x\}$ spans $P_3(\mathbb{R})$.

3. Determine whether the following sets are linearly independent. If so, prove they are, and if not, explain why not.

   (a) $\{(1, 2, 1, 0), (3, 1, 0, 0), (0, 0, -1, -2), (-1, 3, 3, 2)\} \subset \mathbb{R}^4$

   (b) $\{1 + x + x^3, x^2 + 2x^3, 6 + x^2 + 2x^3, 4 + x\} \subset P_3(\mathbb{R})$

4. Let S be a linearly independent subset of a vector space V. Show that every subset of S is also linearly independent.

5. Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{e}_1) = \mathbf{e}_1 - \mathbf{e}_2$, $T(\mathbf{e}_2) = \mathbf{e}_2 - \mathbf{e}_3$ and $T(\mathbf{e}_3) = \mathbf{e}_3 - \mathbf{e}_1$.

   (a) What is the matrix of T with respect to the standard basis?

   (b) What is $T(x_1, x_2, x_3)$ for an arbitrary vector $\mathbf{x} = (x_1, x_2, x_3)$?

   (c) Find a basis for the kernel of T. What is the dimension of $Ker(T)$?

   (d) Find a basis for the image of T. What is the dimension of $Im(T)$?

   (e) What is the matrix of T with respect to the basis $\alpha = \{(1, 1, 1), (1, 0, 2), (-1, -1, 4)\}$ in the domain and the basis $\beta = \{(1, -1, 0), (2, 0, 1), (0, 0, 1)\}$ in the target.

6. Define a mapping $T : \mathbb{R}^n \to \mathbb{R}$ by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{a} \rangle = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ (for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Show that T is a linear transformation. (Thus, the inner product is linear in the first component.)

7. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $S(a_1, a_2) = (2a_1 + 3a_2, -a_1 + ax_2)$ and $T : \mathbb{R}^2 \to P_2(\mathbb{R})$ be given by $T(a_1, a_2) = a_1x^2 - 3a_2x + 5a_1 - 3a_2$. Let α be the standard basis for $\mathbb{R}^2$ and $\beta = \{1, x, x^2\}$. Find $[TS]_\alpha^\beta$.

8. Let $T : \mathbb{R}^7 \to \mathbb{R}^4$ be a linear transformation. Without using the dimension theorem, prove that $\dim(\ker(T)) \geq 3$. (Hint: consider the matrix of T with respect to the standard bases of $\mathbb{R}^7$ and $\mathbb{R}^4$.)

9. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ whose matrix with respect with the standard basis is

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 3 & 0 & -2 \\ 0 & 0 & 3 & 0 \\ 6 & 0 & 0 & -1 \end{bmatrix}.$$

   (a) Find the eigenvalues for T.

   (b) For each eigenvalue, find a basis for the corresponding eigenspace.

   (c) Is T diagonalizable? If yes, give a basis of $\mathbb{R}^3$ consisting of eigenvectors for T. If not, explain why not.

10. Let V be a vector space and let $T : V \to V$ be a linear transformation with the property that $T^2 = I$, i.e. $T \circ T$ is the identity transformation.

   (a) Show that if λ is an eigenvalue of T, then $\lambda = 1$ or $\lambda = -1$.

   (b) Show that the eigenspaces satisfy $E_1 \cap E_{-1} = \{\mathbf{0}\}$.

   (c) Assume that $V = E_1 + E_{-1}$ (i.e. every vector in V can be written as the sum of a vector in E_1 and $_{-1}$. Does this mean that T must be diagonalizable? Explain.