College of the Holy Cross, Spring 2009
Math 244, Midterm 3
Tuesday, April 21

1. Consider the linear transformation $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p) = p' + p''$.

(a) [5 points] For $\alpha = \{1, x, x^2, x^3\}$ basis for $P_3(\mathbb{R})$ and $\beta = \{1, x, x^2\}$ basis for $P_2(\mathbb{R})$, find the matrix of T with respect to α and β, *i.e.*, find $[T]_\alpha^\beta$.

$$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x^2) = 2x + 2 = 2\cdot 1 + 2\cdot x + 0\cdot x^2$$
$$T(x^3) = 3x^2 + 6x = 0\cdot 1 + 6\cdot x + 3\cdot x^2$$

$$[T]_\alpha^\beta = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b) [5 points] What is the dimension of Ker(T)? Find a basis for Ker(T).

Need to solve the system given by the augmented matrix

$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

x_1 is the only free var, so $\dim(\mathrm{Ker}(T)) = 1$.

Have $x_2 = x_3 = x_4 = 0$, so a basis for Ker(T) consists of the vector corresponding to $(1,0,0,0)$, which is 1.

(thought of as a polynomial.)

(c) [5 points] What is the dimension of Im(T)? Find a basis for Im(T)

In the echelon form of $[T]_{\gamma}^{\beta}$ there are 3 basic. var.s, so $\dim(\mathrm{Im}(T)) = 3$. A basis consists of the vectors corresponding to the col.s containing the basic variables, namely

$$\{1,\ 2+2x,\ 6x+3x^2\}$$

One can also use the dimension thm. to deduce that $\dim(\mathrm{Im}(T)) = 3$.

(d) [5 points] Is T injective? Explain your answer.

No, since $\dim(\mathrm{Ker}(T)) > 0$.

(e) [5 points] Is T surjective? Explain your answer.

Yes, since $\dim(\mathrm{Im}(T)) = 3$ and $\dim(P_2(\mathbb{R})) = 3$. Thus $\mathrm{Im}(T) = P_2(\mathbb{R})$, and T is surjective.

2. [10 points] Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be given by rotation through a 90 degree angle, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have the matrix

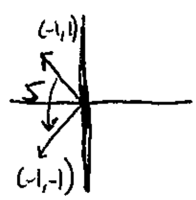
$$A = \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}$$

with respect to the standard basis of $\mathbb{R}^2$. Find the matrix for ST with respect to the standard basis of $\mathbb{R}^2$.

One method is to determine what ST does to the vectors in the standard basis for $\mathbb{R}^2$.

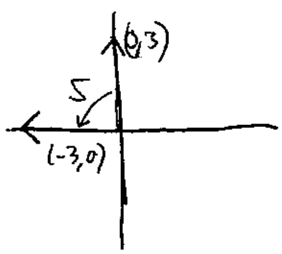
To find $ST(1,0)$, note that $T(1,0)$ is $\begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

so we need to find $S(-1,1)$.



Thus $ST(1,0) = (-1,-1)$. So the 1st col. of our matrix is $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

To find $ST(0,1)$, note that $T(0,1)$ is $\begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$



so $ST(0,1) = (-3,0)$.

Thus $[ST]_\gamma^\gamma = \begin{bmatrix} -1 & -3 \\ -1 & 0 \end{bmatrix}$

std. basis

3. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix with respect with the standard basis is $A = \begin{bmatrix} 2 & 0 & 3 \\ 2 & 1 & 6 \\ 1 & 2 & 8 \end{bmatrix}$.

(a) [10 points] Is T invertible? If so, what is the matrix of T^{-1} with respect to the standard basis?

To see if T invrtble, need to check if is inj. and surj. We will do this as part of finding T^{-1}:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \\ 1 & 2 & 8 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 0 & -1 & -3 & 1 & -1 & 0 \\ 0 & -3 & -10 & 0 & 1 & -2 \\ 1 & 2 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & 4 & -2 \\ 1 & 2 & 8 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 3 & -4 & 2 \\ 0 & 1 & 0 & -10 & 13 & -6 \\ 1 & 2 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 6 & -3 \\ 0 & 1 & 0 & -10 & 13 & -6 \\ 0 & 0 & 1 & 3 & -4 & 2 \end{array}\right]$$

The left part of this matrix shows the echelon form for A has 3 basic var.s and no free var.s, so T is invertible. Moreover, $[T^{-1}]_\gamma^\gamma = \left([T]_\gamma^\gamma\right)^{-1}$, so the matrix for T^{-1} is $\begin{bmatrix} -4 & 6 & -3 \\ -10 & 13 & -6 \\ 3 & -4 & 2 \end{bmatrix}$

(b) [10 points] Suppose that α is any basis for $\mathbb{R}^3$. Prove that the determinant of $[T]_\alpha^\alpha$ is 1. [Hint: $[T]_\alpha^\alpha = Q^{-1}AQ$ for some matrix Q (in fact, $Q = [I]_\alpha^\beta$, where β is the standard basis).]

The matrix A here represents the matrix from part (a).

$$\det([T]_\alpha^\alpha) = \det(Q^{-1}AQ)$$

$$= \det(Q^{-1}) \cdot \det(A) \cdot \det(Q)$$

$$= \frac{1}{\det(Q)} \cdot \det(A) \cdot \det(Q)$$

$$= \det(A) \qquad \text{[since mult. of real \#s commutes]}$$

But expanding along the first row, get

$$\det(A) = 2\cdot(8-12) + 3(4-1)$$

$$= -8 + 9$$

$$= \underline{1}$$

4. [10 points] Let V be a finite-dimensional vector space, and suppose that $T : V \to V$ is a surjective linear transformation. Prove that if $\{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis for V, then $\{T(\vec{v_1}), \ldots, T(\vec{v_n})\}$ is also a basis for V.

We need to show $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is linearly independent and spans V.

To show $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is lin. indep, first note that since $T: V \to V$ and T is surjective, T is also injective (thm. from class). (Prop. 2.4.10)

Now suppose $a_1 T(\vec{v}_1) + \cdots + a_n T(\vec{v}_n) = \vec{0}$. By linearity of T, $T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = \vec{0}$. Since T injective, $\ker(T) = \{\vec{0}\}$, so $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$. But $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is lin. indep. (since it's a basis for V), so $a_1 = a_2 = \cdots = a_n = 0$. This shows $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ lin. indep.

To show $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ spans V, recall that $\mathrm{Im}(T)$ is spanned by $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ (Prop. 2.3.12). But T is surjective, so $\mathrm{Im}(T) = V$. Thus $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ spans V.

5. (a) [8 points] Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^7$ such that $\dim(\mathrm{Im}(T)) = 2\dim(\mathrm{Ker}(T))$? If your answer is yes, give one such transformation. If your answer is not, explain why not.

By the dim. thm, $\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) = \dim(\mathbb{R}^3) = 3$

If $\dim(\mathrm{Im}(T)) = 2\dim(\mathrm{Ker}(T))$, then we have

$3\dim(\mathrm{Ker}(T)) = 3$, so $\dim(\mathrm{Ker}(T)) = 1$

An example of such a transformation is

$T(x_1, x_2, x_3) = (x_1, x_2, 0, 0, 0, 0, 0)$

whose matrix $[T]_{\alpha}^{\beta}$ = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

std. bases

(b) [7 points] What are the possible dimensions for $\mathrm{Im}(T)$ if $T : \mathbb{R}^3 \to \mathbb{R}^7$ is a linear transformation? What are the possible dimensions for $\mathrm{Ker}(T)$

Since $\dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T)) = 3$,

$\dim(\mathrm{Im}(T))$ can be $0, 1, 2,$ or 3

$\dim(\mathrm{Ker}(T))$ can be $0, 1, 2,$ or 3

6. [10 points]For what values of a is the matrix $\begin{bmatrix} a & -a & a \\ -a & -a & a \\ 1 & 1 & a \end{bmatrix}$ invertible?

Need determinant of the matrix to be non-zero.

expand along 3rd col:

$$(-1)^{1+3} a \left(-a - (-a)\right) + (-1)^{2+3} a\left(a - (-a)\right) + (-1)^{3+3} a \left(-a^2 - a^2\right)$$

$$= \quad 0 \quad + \quad -a(2a) \quad + \quad a(-2a^2)$$

$$= \quad -2a^2 - 2a^3$$

$$= \quad -2a^2(1+a)$$

Thus if $a \neq 0$ and $a \neq -1$, the matrix is invertible.

7. [10 points] Suppose V and W are finite dimensional vector spaces and $\dim(V) \leq \dim(W)$. Show that there exists an injective linear transformation from V to W. [Hint: say specifically what the transformation would be in terms of its action on a basis for V.]

Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V

Let $\{\vec{w}_1, \dots, \vec{w}_k\}$ be a basis for W

Note $n \leq k$ since $\dim(V) \leq \dim(W)$

Let $T(\vec{v}_1) = \vec{w}_1$
$T(\vec{v}_2) = \vec{w}_2$
$\vdots$
$T(\vec{v}_n) = \vec{w}_n$

Then T is injective. Proof: If $T(\vec{v}) = \vec{0}$, write $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$. Then $T(\vec{v}) = a_1 T(\vec{v}_1) + \dots + a_n T(\vec{v}_n)$
$= a_1\vec{w}_1 + \dots + a_n\vec{w}_n$.

So $\vec{0} = a_1\vec{w}_1 + \dots + a_n\vec{w}_n + 0\vec{w}_{n+1} + \dots + 0\vec{w}_k$

Thus $a_1 = a_2 = \dots = a_n = 0$ since $\{\vec{w}_1, \dots, \vec{w}_k\}$ is linearly indep.
Hence $\vec{v} = 0\vec{v}_1 + \dots + 0\vec{v}_n = \vec{0}$. We've shown $\operatorname{Ker}(T) = \{\vec{0}\}$, so T injective.

Note: if $\alpha = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\beta = \{\vec{w}_1, \dots, \vec{w}_k\}$, then

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \rightarrow n^{th} \text{ row} \\ \\ \\ \end{array}$$

This matrix is in ech. form and has no free var.s, which gives another pf. that T is injective.