College of the Holy Cross, Fall 2008
Math 243, Practice Midterm 2

The problems on the actual exam will be somewhat similar to the problems here but some problems could look very different (in form or degree of difficulty) from the problems here. This practice exam is a bit longer than the actual exam will be. Working on this practice exam should **NOT** be your only preparation for the exam. Make sure you understand how to do all the problems on the homework. The second exam covers sections 1.4, 1.5, 1.6, 1.7, and 2.2.

1. Let $*$ be the binary operation on the set of integers $\mathbb{Z}$ defined by:

$$x * y = x + xy + y - 2.$$

   Determine if $*$ is:

   (a) associative

   (b) commutative

   (c) if there is an identity element for $*$.

   In each case prove your conclusion.

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ \dfrac{x-1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

   Find a right inverse for f or explain why none exists. Find a left inverse for f or explain why none exists.

3. Find x and y such that the following matrix product gives the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & x & 2 \\ 2 & -2 & y \end{bmatrix}$$

4. Let G be the subset of $M_{2\times 2}(\mathbb{R})$ given by

$$G = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R} \right\}.$$

   (a) Show that G is closed under matrix multiplication.

(b) Show that the identity matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is in G.

(c) For each matrix $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ in G, find a matrix B in G such that $AB = I_2$.

5. Use mathematical induction to prove that the assertion P_n holds for every positive integer n, where

$$P_n : \qquad \frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}.$$

6. Consider the set $\mathcal{P}(A) - \emptyset$ of all non-empty subsets of $A = \{1, 2, 3, 4, 5\}$. Let R be the relation on $\mathcal{P}(A) - \emptyset$ defined by

$$XRY \qquad \text{if and only if} \qquad X \subseteq Y.$$

Determine whether R is reflexive, symmetric, or transitive.

7. Use mathematical induction to show that that if a set S has n distinct elements, then S has 2^n distinct subsets.