College of the Holy Cross, Fall 2008
Math 243, Practice Midterm 1

1. Negate the following statement: *There is a politician who is either honest or not trustworthy.* Do **not** use "There is no politician..."

   All politicians are both trustworthy and not honest.

2. Write the converse, inverse and contrapositive of the following implication, and determine the truth or falsity of each statement (including the implication). You do **not** need to prove the truth of falsity of the statement.
   Implication: *If x and y are integers divisible by 3, then $x + y$ is an integer divisible by 3.*

   The implication is true.

   The converse is *If $x + y$ is an integer divisible by 3, then x and y are integers divisible by 3.* This is false, as is shown by $2 + 1 = 3$.

   The inverse is *If either x or y is not divisible by 3, then $x + y$ is not divisible by 3.* This is false, as is shown by $2 + 1 = 3$.

   The contrapositive is *If $x + y$ is not divisible by 3, then either x or y is not divisible by 3.* This is true.

3. Let A and B be sets. Prove that if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

   Suppose that $A \subseteq B$. This means that every element of A is also an element of B. We now wish to show that $A \cap C \subseteq B \cap C$. So we use our standard technique of taking an element $x \in A \cap C$ and showing that it is in $B \cap C$:

   Let $x \in A \cap C$. Thus $x \in A$ and $x \in C$. Since every element of A is also in B, this gives $x \in B$ and $x \in C$. Thus $x \in B \cap C$, and the proof is complete.

4. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be given by

   $$f(x, y) = (x + y, 1).$$

   Is f onto? Is f one-to-one? In each case prove your conclusion.

   1) No, f is not onto. The reason is that the second coordinate of each element of the range of f is always 1. Thus, for instance, $(0, 2)$ is not in the range of f, or put another way, $f(x, y) \neq (0, 2)$ for all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$.

2) No, f is not one-to-one. For instance, $f(0,0)$ and $f(1,-1)$ are both equal to $(0,1)$, even though $(0,0) \neq (1,-1)$.

5. Let $A = \{1,2,3,4,5,6,7,8,9\}$, and let $\mathcal{P}(A)$ be the power set of A. Consider the map $f : A \to \mathcal{P}(A)$ that sends each $x \in A$ to the set $A - \{x\}$. Thus

$$f(1) = \{2,3,4,5,6,7,8,9\}.$$

Is f onto? Is f one-to-one? In each case prove your conclusion.

1) No, f is not onto. Every element in the range of f is a subset of A with nine elements. Thus no element of A maps to any set not having nine elements, such as $\{1,2\}$.

2) Yes, f is one-to-one. One way to see this is as follows: assume that $f(a_1) = f(a_2)$. This means $A - \{a_1\} = A - \{a_2\}$. Thus $A - (A - \{a_1\}) = A - (A - \{a_2\})$. But $A - (A - \{a_1\}) = \{a_1\}$ and $A - (A - \{a_2\}) = \{a_2\}$. Hence $a_1 = a_2$.

Another way to argue is to note that if $x \neq y$ then $x \notin f(x)$ (this makes sense since here $f(x)$ is an element of $\mathcal{P}(A)$, and thus is a set) but $x \in f(y)$. This means that $f(x) \neq f(y)$, since two sets are equal if and only if they have the same elements. This shows that f is one-to-one.

Finally, since there are only nine sets in the image $f(A)$ of A, you could write them all out and note that they are all different.

6. Prove or disprove the following statement: for any sets A, B, and C, $A \cup B = A \cup C$ implies that $B = C$.

False. Let $A = \{1,2,3\}$, $B = \{1,2\}$, and $C = \{3\}$. Then $A \cup B = A \cup C$, since both are $\{1,2,3\}$. But $B \neq C$.

7. Compute $\mathcal{P}(\mathcal{P}(\emptyset))$, where $\emptyset$ denotes the empty set.

First, the only subset of the empty set is the empty set, so we have $\mathcal{P}(\emptyset) = \{\emptyset\}$. Note that this is different from the empty set – it is a set that has one element (namely the empty set), whereas the empty set has no elements. Now, $\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}\{\emptyset\}$. Since $\{\emptyset\}$ has only one element, its only subsets are the empty set and the set consisting of the one element. This is

$$\{\emptyset, \{\emptyset\}\}.$$