College of the Holy Cross, Fall 2008
Math 243, Practice Final

The final exam is cumulative, and so will cover everything we've discussed this semester: Logic, sections 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 2.2, 2.3, 2.4, 2.5, 2.6, 3.1, 3.2, 3.5, and parts of 3.3 and 3.4. You should know how to state and to use the theorems covered in class, and how to do all problems on the homework. Part of the final exam will consist of proving one or two theorems from the list below. The rest of the exam will consist of problems similar to those on the midterms and homework.

Theorems for the Final Exam

Theorem 2.14: If a and b are relatively prime and $a \mid bc$, then $a \mid c$.

Theorem 2.16: Euclid's Lemma

Theorem 2.19: Euclid's Theorem on the Infinitude of the Primes

Theorem 3.28: Images of identities and inverses under homomorphisms

As with the practice midterms, the problems on the actual exam will be somewhat similar to the problems here but some problems could look very different (in form or degree of difficulty) from the problems here. This practice exam is also somewhat longer than the actual final will be – you can expect about 8-9 problems on the final. However, working on this practice exam should **NOT** be your only preparation for the exam.

On the final, you will be given the following multiplication tables, for S_3 and a group called D_4, which has eight elements:

Multiplication table for S_3:

$\circ$	e	ρ	ρ^2	σ	γ	δ
e	e	ρ	ρ^2	σ	γ	δ
ρ	ρ	ρ^2	e	γ	δ	σ
ρ^2	ρ^2	e	ρ	δ	σ	γ
σ	σ	δ	γ	e	ρ^2	ρ
γ	γ	σ	δ	ρ	e	ρ^2
δ	δ	γ	σ	ρ^2	ρ	e

Multiplication table for D_4:

$\circ$	e	α	α^2	α^3	β	γ	δ	θ
e	e	α	α^2	α^3	β	γ	δ	θ
α	α	α^2	α^3	e	γ	δ	θ	β
α^2	α^2	α^3	e	α	δ	θ	β	γ
α^3	α^3	e	α	α^2	θ	β	γ	δ
β	β	θ	δ	γ	e	α^3	α^2	α
γ	γ	β	θ	δ	α	e	α^3	α^2
δ	δ	γ	β	θ	α^2	α	e	α^3
θ	θ	δ	γ	β	α^3	α^2	α	e

1. Consider the following two maps $\mathbb{Z} \to \mathbb{Z}$:

$$f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases} \qquad g(x) = 2x$$

   Find $f \circ g$ and $g \circ f$, and say whether each one is one-to-one and onto. You only need to justify negative answers.

2. Define a relation on $\mathbb{Z}$ as follows: xRy if and only if $x + 3y$ is a multiple of 4. Show that R is an equivalence relation, and find the equivalence class of 1.

3. Suppose that a and b are integers with $(a, b) = 1$. Prove that $(a, b^2) = 1$.

4. (a) Find the multiplicative inverse of [33] in $\mathbb{Z}_{58}$.

   (b) Use prime factorizations to find the GCD of 192 and 240.

5. Use induction to show that 8 is a factor of $9^n - 1$ for all $n \in \mathbb{Z}^+$

6. Let G be the following subset of the set of invertible 2×2 matrices

$$G = \left\{ A \in M_2(\mathbb{R}) \mid A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, a \neq 0, c \neq 0 \right\}.$$

   Show that G forms a group under matrix multiplication.

7. Consider the set I of all elements in $\mathbb{Z}_{12}$ that have multiplicative inverses. You may take it for granted that this set forms a group under multiplication.

   (i) List the elements of I.

   (ii) Compute the order of each element of I.

   (iii) Is I a cyclic group? Justify your answer.

   (iv) What is the multiplicative inverse of [7]?

8. Suppose G is a group and let $a, b \in G$. Prove that if $a \in \langle b \rangle$, then $\langle a \rangle \subseteq \langle b \rangle$.

9. Let G be an abelian group. An element $a \in G$ with finite order is called a *torsion* element. Prove that the set of all torsion elements in G is a subgroup of G.

10. Let G be a group and let g be a *fixed* element in G. Consider the following subset of G.

$$H_\alpha = \{y \in G \mid y = bgxg^{-1}, \text{ for some } x \in G\}.$$

   (a) For $G = D_4$, $g = \beta$, find H_g.

   (b) Prove that H_g is a subgroup of G.

11. Let G and G' be groups and let e be the identity of G and e' be the identity of G'. Let $\phi : G \to G'$ be a homomorphism. Recall that Ker $\phi = \{x \in G \mid \phi(x) = e'\}$. Show that ϕ is one-to-one if and only if Ker $\phi = \{e\}$.

12. Let $G = \mathbb{Z}$, with operation addition, and let $G' = G$. Consider the map $f : G \to G'$ given by $f(x) = -x$. Prove that f is an isomorphism.