Math 243: Notes on Induction

The homework problems on induction involve not just getting the equations to work out in the third step, but also writing a correct and fully explained argument. This handout describes what's involved in doing this, and gives some additional details about how induction problems were graded on the HW and how they'll be graded on exams.

- The *base case* of an induction argument consists of checking that the statement holds when $n = 1$ (or $n = a$ for generalized induction). It's an important part of the argument, as HW problem # 40 shows. Everyone did a good job of checking the base case on the homework.
- The *inductive hypothesis* (or inductive assumption), is the assumption that the statement holds for $n = k$ (or for complete induction that it holds for m where $a \leq m \leq k$). Whenever you write a proof by induction, you should write out the inductive hypothesis. On an exam, not doing this would deduct 3 points out of 10, even if the rest is right.
- The last step is showing that your statement holds for $n = k + 1$. It is always a good idea to write out the statement for $n = k+1$, so that you're clear on what you're trying to show. You should then take the left-hand side of this equation or inequality, apply the inductive hypothesis to it, and if necessary use other manipulations that wind up giving you the right-hand side. In particular, you should make clear where you use the inductive hypothesis. Not doing so will cost 1 point out of 10 on an exam problem (on the HW I took off 0.5 per problem for this, but I stopped once the total deduction reached one point for the set).

  One common mistake is to write the equation or inequality for $n = k + 1$ and then to begin manipulating this equation, winding up with the last step being something that is clearly true. As an example, suppose your inductive step is $1+2+\cdots+k = k(k+1)/2$. You then might write

  $$\text{Want to show:} \qquad 1 + 2 + \cdots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}.$$

  $$\begin{aligned} \frac{k(k+1)}{2} + (k+1) &= \frac{(k+1)(k+2)}{2} && \text{Inductive hypothesis} \\ \frac{k}{2} + 1 &= \frac{k+2}{2} && \text{Divide by } (k+1) \\ \frac{k+2}{2} &= \frac{k+2}{2} \end{aligned}$$

  This is fallacious reasoning. This is because by manipulating the equation that we want to show, we are assuming that it is in fact true. But the whole point is to *prove* that this equation is true. As an illustration, let's use this kind of reasoning to show that $-1 = 1$:

  $$\text{Assume that } -1 = 1$$

$$\text{Then } (-1)^2 = 1^2$$

$$\text{So } 1 = 1$$

$$\text{But that's clearly true. So we've shown } -1 = 1.$$

The error is that at the very beginning we assumed what we were trying to prove. So please avoid doing this. *One way to spot this error is the following: if your final step is that some expression equals itself, it means that you've made this error.* On an exam, if you argue by assuming what you want to prove, you will get at most about half credit for the problem. On the HW, I was more lenient: I took off one point per problem, but I stopped once the total deduction reached four points for the set.

What should have been done in the first example above is a chain of equalities turning the left-hand side of the desired equation into the right-hand side:

$$\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{Inductive hypothesis} \\
&= (k+1)\left(\frac{k}{2} + 1\right) \\
&= (k+1)\left(\frac{k+2}{2}\right) \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Sometimes it is helpful to work on each side of the desired equation or inequality separately, and get them to meet in the middle. In the above example, we could have argued like this:

$$\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{Inductive hypothesis} \\
&= (k+1)\left(\frac{k}{2} + 1\right)
\end{aligned}$$

Also note that

$$\frac{(k+1)(k+2)}{2} = (k+1)\left(\frac{k+2}{2}\right) = (k+1)\left(\frac{k}{2} + 1\right)$$

It follows that $1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$, which is what we wanted to show.