College of the Holy Cross, Fall 2008
Math 243, Midterm 1
Tuesday, September 30

1. [10 points] Negate the following statement. *There exist real numbers x and y such that $x^2 + y^2 < 0$.* Do **not** use "There do not exist real numbers..."

   For all real numbers x and y, $x^2 + y^2 \geq 0$.

2. [20 points] Write the converse, inverse and contrapositive of the following implication, and determine the truth or falsity of each statement (including the implication). You do **not** need to prove the truth of falsity of the statement.
   Implication: *If $n = 3k + 1$ for some integer k, then n is not divisible by 3.*

   The implication is true. Indeed, if $n = 3k + 1$ for some integer k then n has remainder 1 when divided by three, and thus is not divisible by three.

   Converse: if n is not divisible by 3 then $n = 3k + 1$ for some integer k. This is false, as is shown by $n = 2$, which is not divisible by 3 but has remainder 2 when divided by 3, and thus is not of the form $n = 3k + 1$.

   Inverse: if $n \neq 3k + 1$ for some integer k, then n is divisible by 3. False, again shown by $n = 2$.

   Contrapositive: if n is divisible by 3 then $n \neq 3k + 1$ for some integer k. True, since the original implication is true and the contrapositive has the same truth value. Alternatively, if 3 divides n then n cannot have remainder 1 when divided by 3, and so is not of the form $3k + 1$.

3. [15 points] Let A and B be sets. Prove that $(A - B) \cup (A \cap B) = A$.

To show that $(A - B) \cup (A \cap B) = A$ it's enough to show that $(A - B) \cup (A \cap B) \subseteq A$ and $(A - B) \cup (A \cap B) \supseteq A$

First we show $(A - B) \cup (A \cap B) \subseteq A$. To do this, assume $x \in (A - B) \cup (A \cap B)$.

Thus $x \in (A - B)$ or $x \in (A \cap B)$.

Hence either $x \in A$ and $x \notin B$, or $x \in A$ and $x \in B$. We now have two cases:

if $x \in A$ and $x \notin B$, then clearly $x \in A$.

if $x \in A$ and $x \in B$, then also $x \in A$.

Therefore $x \in A$ in either case. We've shown $(A - B) \cup (A \cap B) \subseteq A$.

We now show $(A - B) \cup (A \cap B) \supseteq A$. Suppose that $x \in A$. There are two cases: $x \notin B$ and $x \in B$.

if $x \notin B$, then we have $x \in A - B$, since $A - B = \{x : x \in A, x \notin B\}$.

if $x \in B$, then we have $x \in A \cap B$.

Thus in either case we have $x \in (A-B) \cup (A \cap B)$. We've shown $(A-B) \cup (A \cap B) \supseteq A$, completing the proof.

4. Let $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be given by

$$f(x) = (x^2, x)$$

(a) [5 points] For $S = \{-2, 0, 2\}$ find $f(S)$.

$f(-2) = (4, -2)$, $f(0) = (0, 0)$, and $f(2) = (4, 2)$, so

$$f(S) = \{(-4, 2), (0, 0), (4, 2)\}.$$

(b) [5 points] For $T = \{(1, 1), (2, 2), (9, -3)\}$, find $f^{-1}(T)$.

The only element mapping to $(1, 1)$ is 1, and the only element mapping to $(9, -3)$ is -3. Nothing maps to $(2, 2)$ since it is not of the form (x^2, x) for any $x \in \mathbb{Z}$. Thus

$$f^{-1}(T) = \{1, -3\}.$$

(c) [5 points] For $T = \{(0, 1), (4, 4)\}$, find $f^{-1}(T)$.

Neither $(0, 1)$ or $(4, 4)$ is of the form (x^2, x) for $x \in \mathbb{Z}$, so $f^{-1}(T) = \emptyset$.

5. [15 points] Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ \dfrac{x-1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

Is f onto? Is f one-to-one? In each case prove your conclusion.

Yes, f is onto. To get an idea of why, we can compute some values of f:

$$\begin{aligned} f(-3) &= -2 \\ f(-2) &= -1 \\ f(-1) &= -1 \\ f(0) &= 1 \\ f(1) &= 0 \\ f(2) &= 3 \\ f(3) &= 1 \\ f(4) &= 5 \\ f(5) &= 2 \end{aligned}$$

To show f is onto, we take $b \in \mathbb{Z}$ and show that there is $a \in \mathbb{Z}$ with $f(a) = b$. In other words, we solve the equation $f(a) = b$ for a.

Here's one way to proceed: if a is even, then we are solving the equation $a + 1 = b$, which means $a = b - 1$. Now since a is even and $a = b - 1$, $b - 1$ must be even, so b must be odd. Thus in the case of b odd, we have $f(b - 1) = b$. (Check this with the values given above.)

On the other hand, if a is odd, then we are solving the equation $(a - 1)/2 = b$, which means $a = 2b + 1$. Note that regardless of the value of b, a will be odd. This shows that $f(2b + 1) = b$ for *any* b, and in particular for b even. (Check this with the values given above.)

Hence no matter whether b is odd or even, it is the image of some element of $\mathbb{Z}$. Thus f is surjective.

Alternatively, the analysis of a odd shows directly that $f(2b+1) = b$ for all $b \in \mathbb{Z}$, and thus b is the image of some element of $\mathbb{Z}$. Hence f is surjective.

No, f is not one-to-one. From the values above, we have $f(-1) = f(-2)$.

6. Let $A = \{1, 2, 3\}$, and let $\mathcal{P}(A)$ be the power set of A.

   (a) [5 points] Give an example of a map $A \to \mathcal{P}(A)$ that is injective but not surjective. You don't need to justify your answer. Note: Just say which elements of $\mathcal{P}(A)$ $1, 2$, and 3 map to.

   Note that
   $$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$$
   If we put $f(1) = \emptyset$, $f(2) = \{1, 2\}$ and $f(3) = A$ then no two different elements have the same images, so f is one-to-one. Clearly f is not surjective, since for instance nothing maps to $\{1\}$.

   (b) [5 points] Give an example of a map $A \to \mathcal{P}(A)$ that is neither injective nor surjective. You don't need to justify your answer.

   Let $f(1)$, $f(2)$, and $f(3)$ all be $\emptyset$.

   (c) [5 points] Is it possible to find a map $A \to \mathcal{P}(A)$ that is surjective? Why or why not?

No, it's not possible. The reason is that $\mathcal{P}(A)$ has 8 elements, and A only has 3 elements. Since a mapping must map each element of the domain to exactly one element of the codomain (that's the definition of a mapping), the range of A can have at most 3 elements. Thus there will always be elements of $\mathcal{P}(A)$ that are not in the range of A, so no mapping $A \to \mathcal{P}(A)$ can be surjective.

A similar argument shows that if A and B are any finite sets with A having strictly fewer elements than B, then there cannot be a surjective mapping $A \to B$.

7. Let $A = \{1\}$.

(a) [5 points] How many elements does $\mathcal{P}(\mathcal{P}(A))$ have?

If C is a set with n elements, then $\mathcal{P}(C)$ has 2^n elements. Thus $\mathcal{P}(A)$ has 2 elements, and $\mathcal{P}(\mathcal{P}(A))$ has 4 elements.

(b) [5 points] Compute $\mathcal{P}(\mathcal{P}(A))$.

First, we have $\mathcal{P}(A) = \{\emptyset, \{1\}\}$. So

$$\mathcal{P}(\mathcal{P}(A)) = \Big\{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\Big\}.$$

8. *Bonus Problem* [*5 points*]: Find a binary relation $*$ on $\mathbb{Z}$ that is commutative and associative, but does NOT have an identity element. Prove that the relation you find has all these properties.

$x * y = 2xy$ works. Commutativity is obvious. For associativity, a computation shows that $x * (y * z)$ and $(x * y) * z$ both equal $4xyz$.

However, there cannot be an identity element. The reason is that one of the conditions an identity element e would have to satisfy is $x * e = x$ for all $x \in \mathbb{Z}$. But this is the same as

$$2xe = x,$$

which implies $2e = 1$. But twice an integer can never be 1, so no such e lies in $\mathbb{Z}$.