College of the Holy Cross
Math 126, Spring 2009

Prof. Jones
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Exam 3

Name: Solutions

You must explain all your work to receive credit for your answers

These problems are not necessarily arranged in ascending order of difficulty. Work them in an order that will maximize your score. If you need more space, use the back of the page. *Good luck!*

Problem	Score	Problem	Score
1		5	
2		6	
3			
4		Total	

1. (a) [12 points] Verify that the following function is a probability density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{(x+1)^2} & \text{if } x \geq 0 \end{cases}$$

Need to show $\int_{-\infty}^{\infty} f(x)\,dx = 1$

since $f(x) = 0$ for $x < 0$, need $\int_0^{\infty} f(x)\,dx = 1$,

so $\int_0^{\infty} \frac{1}{(x+1)^2}\,dx = 1$.

$$\int_0^{\infty} \frac{1}{(x+1)^2}\,dx = \lim_{z\to\infty} \int_0^{z} \frac{1}{(x+1)^2}\,dx$$

$u = x+1$
$du = dx$

$$= \lim_{z\to\infty} \int_0^{z} \frac{1}{u^2}\,du$$

$$= \lim_{z\to\infty} \int_0^{z} u^{-2}\,du$$

$$= \lim_{z\to\infty} -u^{-1}\Big]_0^{z}$$

$$= \lim_{z\to\infty} -\frac{1}{(x+1)}\Big]_0^{z}$$

$$= \lim_{z\to\infty} -\frac{1}{(z+1)} - -\frac{1}{(0+1)}$$

$$= 0 + 1$$ since $\lim_{z\to\infty} -\frac{1}{(z+1)} = 0$.

(b) [12 points] Suppose the function in part (a) is the probability density function for the lifespan of a car battery (in years). Find the probability that the battery lasts less than four years.

The probability the battery lasts less than four years is

$$\int_0^4 \frac{1}{(x+1)^2}\,dx \overset{\text{using work from part (a)}}{=} -\frac{1}{(x+1)}\Big]_0^4$$

$$= -\frac{1}{4+1} - \left(-\frac{1}{0+1}\right)$$

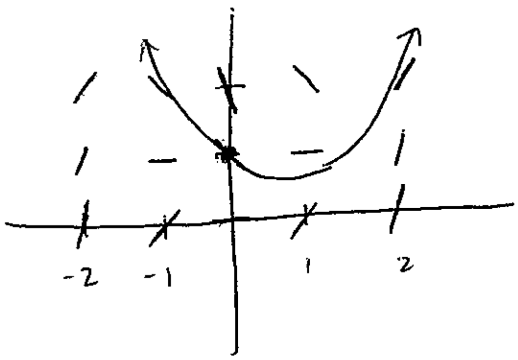
$$= -\frac{1}{5} + 1$$

$$= 4/5$$

2. [12 points] Draw a direction field for the following differential equation:

$$\frac{dy}{dx} = x^2 - y.$$

Your picture only needs to include x-values from -2 to 2 (including -2 and 2) and y-values from 0 to 2 (including 0 and 2). Sketch the solution with $y(0) = 1$.



x	y	x^2-y
0	0	0
0	1	-1
0	2	-2
± 1	0	1
± 1	1	0
± 1	2	-1
± 2	0	4
± 2	1	3
± 2	2	2

3. [12 points] Check that for any value of C, $f(x) = \frac{C}{x}$ is a solution to the differential equation
$$\frac{dy}{dx} = -\frac{y}{x}.$$
[Note: next time, pick a DE that is not separable. Nearly every student tried to find all solutions to this problem via separation of variables.]

We only need to check $y = \frac{C}{x}$ satisfies the diff. eqn:

$$\frac{dy}{dx} = \left(\frac{C}{x}\right)' = \frac{0 - C}{x^2} = -\frac{C}{x^2}$$

$$-\frac{y}{x} = \frac{-\left(\frac{C}{x}\right)}{x} = -\frac{C}{x} \cdot \frac{1}{x} = -\frac{C}{x^2} \quad \checkmark$$

It is also possible (though harder) to do this problem by finding all the solutions via separation of variables:

$$\int \frac{dy}{y} = \int -\frac{dx}{x}$$

$$\ln|y| = -\ln|x| + c$$

$$y = e^{-\ln|x| + c}$$

$$y = e^c \cdot \left(e^{\ln|x|}\right)^{-1}$$

$$y = C \cdot x^{-1}$$

$$y = \frac{C}{x}$$

4. [12 points] Find a solution to the initial value problem

$$\frac{dy}{dx} = \frac{\ln x}{2xy},$$

with $y(1) = 4$

Separate and integrate: $2y\,dy = \frac{\ln x}{x}\,dx$

$\int 2y\,dy = \int \frac{\ln x}{x}\,dx$ $\qquad u = \ln x$, $du = \frac{1}{x}\,dx$

$y^2 = \int u\,du$

$y^2 = \frac{1}{2}u^2 + C$

$y^2 = \frac{(\ln x)^2}{2} + C$

$y = \sqrt{\frac{(\ln x)^2}{2} + C}$

When $x=1$, $y=4$:

$4 = \sqrt{\frac{(\ln 1)^2}{2} + C}$

$4 = \sqrt{C}$

$16 = C$

$\boxed{y = \sqrt{\frac{(\ln x)^2}{2} + 16}}$

5. A hedgehog population grows at a rate proportional to its size. Initially there are 10 hedgehogs, and after one year the population has quadrupled to 40 hedgehogs.

   (a) [12 points] Find an expression for the number of hedgehogs after t years.

Since the population grows at a rate proportional to the size, the population fcn will be exponential: $P = Ce^{kt}$.

Since $P(0) = 10$, $10 = Ce^0 = C$

since $P(1) = 40$, $40 = 10e^{k \cdot 1}$, so $e^k = 4$,

so $k = \ln 4$.

Thus $P(t) = 10e^{(\ln 4)t} = 10 \cdot 4^t$

   (b) [8 points] When will the population reach 1000 hedgehogs?

Need to solve

$$1000 = 10 \cdot 4^t$$
$$100 = 4^t$$
$$100 = e^{(\ln 4)t}$$
$$\ln 100 = (\ln 4)t$$
$$\frac{\ln 100}{\ln 4} = t$$

$$t \approx 3.32$$

6. A muskrat population at time t (in years) is given by the function $P(t)$, which obeys this differential equation:

$$\frac{dP}{dt} = 4\sqrt{P}\cos t.$$

(a) [12 points] Suppose that there are initially 25 muskrats. Find an expression for the number of muskrats after t years (that is, find $P(t)$).

Find P by separating the variables:

$$\frac{dP}{\sqrt{P}} = 4\cos t \, dt$$

$$\int \frac{1}{\sqrt{P}} \, dP = 4\int \cos t \, dt$$

$$\int P^{-1/2} \, dP = 4\sin t + C$$

$$2P^{1/2} = 4\sin t + C$$

$$P^{1/2} = 2\sin t + C$$

$$P = (2\sin t + C)^2$$

When $t=0$, $P=25$, so

$$25 = (2\sin 0 + C)^2$$

$$25 = C^2$$

$$5 = C$$

$$\boxed{P(t) = (2\sin t + 5)^2}$$

(b) [8 points] Will the population ever reach 50 muskrats?

No. since $\sin t \leq 1$ for all values of t, the largest $P(t)$ will ever get is $(2+5)^2$, or 49.

This population oscillates between 9 and 49.