College of the Holy Cross
Math 126, Spring 2009

Prof. Jones
March 19, 2009

Exam 2 Solutions

Name:

You must explain all your work to receive credit for your answers

These problems are not necessarily arranged in ascending order of difficulty. Work them in an order that will maximize your score. If you need more space, use the back of the page. *Good luck!*

Problem	Score	Problem	Score
1		5	
2		6	
3			
4		Total	

1. Suppose the demand function for coffee mugs at a local supermarket is $p(x) = 100/(10+x)$.

   (a) [12 points] Find the consumer surplus when the price of a coffee mug is $5. When $p = 5$, we can find the production level x by plugging 5 in for p and solving

   $5 = 100/(10+x)$, which gives $x = 10$.

   The consumer surplus is then given by

$$\int_0^{10} \frac{100}{10+x} - 5 \, dx$$

   Setting $u = 10 + x$, and note that $du = dx$. Also, when $x = 0, u = 10$ and when $x = 10$, $u = 20$. So our integral becomes

$$\int_{10}^{20} 100u^{-1} - 5,$$

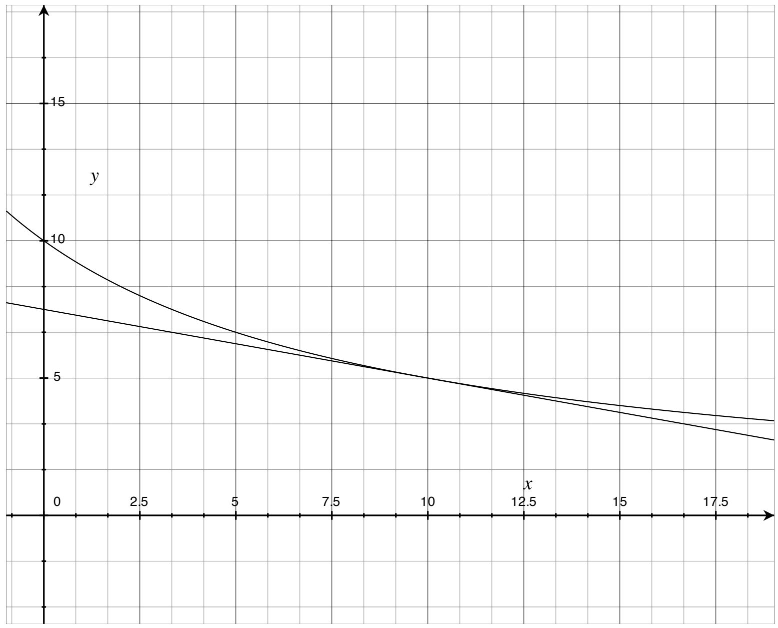
   which is

$$100 \ln u - 5u]_{10}^{20}$$

   and this in turn is

$$100 \ln 20 - 100 - (100 \ln 10 - 50) \approx 69.31 - 50 \approx 19.31.$$

(b) [6 points] Now suppose the demand function for coffee mugs is the line in the following graph (the curve $p(x) = 100/(10 + x)$ is also shown for comparison).



Using only the graph, say whether the consumer surplus for this new linear demand function when the price is \$5 will be less than or greater than your answer for part (a). Explain why this is the case.

The consumer surplus for the new demand function will be less. The reason is that the consumer surplus in this case is the area above the horizontal line $p = 5$ and below the demand function, and the new linear demand function has less such area than the old demand function.

2. [16 points] On a summer's day, the temperature in Trigville is given by

$$T(t) = 70 + 15\sin(t/10),$$

where t is measured in hours and $t = 0$ is at 7 am. Find the average temperature in Trigville from 5 am to noon. Note: make sure your calculator is in *radians* mode: press DRG, then the right arrow, then Enter.

At 5 am, $t = -2$, while at noon, $t = 5$. The average temperature is thus

$$\frac{1}{7}\int_{-2}^{5} 70 + 15\sin(t/10)\,dt.$$

Let $u = t/10$. Then $du = dt/10$, so $dt = 10\,du$. When $t = -2, u = -0.2$, and when $t = 5$, $u = 0.5$. So our integral is

$$\begin{aligned}
\frac{10}{7}\int_{-0.2}^{0.5} 70 + 15\sin(u)\,du. &= \frac{10}{7}\left(70u - 15\cos(u)\right]_{-0.2}^{0.5}\Big) \\
&= \frac{10}{7}(35 - 15\cos(0.5) - (-14 - 15\cos(-0.2))) \\
&\approx \frac{10}{7}(49 + 1.54) \\
&\approx 72.196
\end{aligned}$$

3. [17 points] Determine whether the following integral converges or diverges. If it converges, find its value:

$$\int_e^\infty \frac{1}{x(\ln x)^2}\, dx.$$

$$\int_e^\infty \frac{1}{x(\ln x)^2} dx \quad = \quad \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^2}\, dx$$

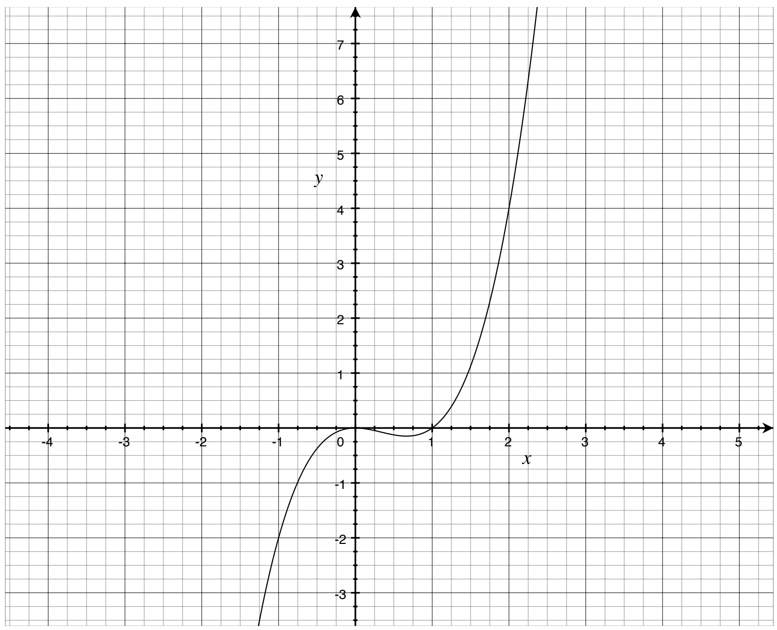
Substitution is the right approach to this integral, since setting $u = \ln x$ gives us $du = 1/x\ dx$, which is already in the integrand. Integration by parts doesn't work, since the product involved is $\frac{1}{x} \cdot \frac{1}{(\ln x)^2}$, and there are no good choices for u and dv.

Making the substitution $u = \ln x$, $du = 1/x\ dx$ gives us

$$\begin{aligned} \lim_{t\to\infty} \int_e^t u^{-2}\, du &= \lim_{t\to\infty} \left(-u^{-1}\Big]_e^t \right) \\ &= \lim_{t\to\infty} \left(-\frac{1}{\ln x}\Big]_e^t \right) \\ &= \lim_{t\to\infty} -\frac{1}{\ln t} + \frac{1}{\ln e} \\ &= \lim_{t\to\infty} -\frac{1}{\ln t} + 1 \end{aligned}$$

Now as $t \to \infty$, $\ln t$ becomes a large positive number (recall the graph of $y = \ln x$. Thus $-\frac{1}{\ln t}$ looks like -1 over a large positive number, which is close to 0. We conclude that $\lim_{t\to\infty} -\frac{1}{\ln t} = 0$, and so our integral converges, with answer 1.

4. [16 points] Suppose the function $f(x)$ has this graph:



Use Simpson's Rule to approximate $\int_{-1}^{2} f(x)\ dx$ with $n = 6$. Note that you will need to estimate some values of $f(x)$.

The endpoints of the six subintervals are $-1, -0.5, 0, 0.5, 1, 1.5$, and 2, and the width of each is $\frac{2-(-1)}{6} = 0.5$. Simpson's Rule says

$$\int_{-1}^{2} f(x)\ dx \approx \frac{0.5}{3}\left(f(-1) + 4f(-0.5) + 2f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)\right).$$

From the graph, we have

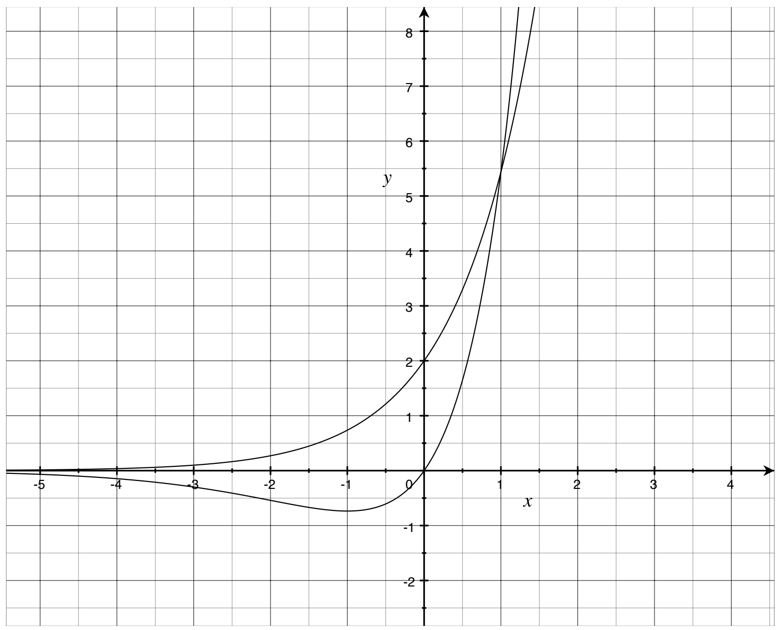
$f(-1) = -2$
$f(-0.5) \approx 0.4$
$f(0) = 0$
$f(0.5) \approx -0.2$
$f(1) = 0$
$f(1.5) \approx 1.2$
$f(2) = 4$

Thus

$$\int_{-1}^{2} f(x)\ dx \approx \frac{1}{6}\left(-2 - 1.6 + 0 - 0.8 + 0 + 4.8 + 4\right),$$

and this comes out to 4.4/6, or about 0.73.

5. [16 points] Find the area bounded by the curves $y = 2xe^x$, $y = 2e^x$, and $x = 0$.



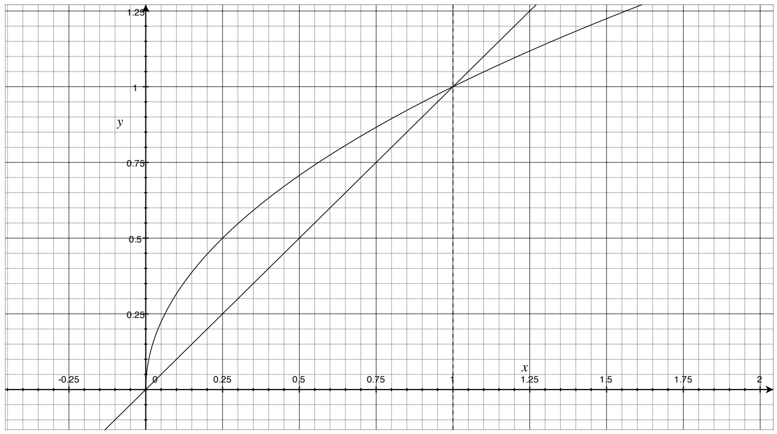
The area in question is that between the two curves and to the right of the y-axis. It has as its top function $y = 2e^x$ and as its bottom function $y = 2xe^x$, and has x-values from 0 to 1. Thus the area is

$$\int_0^1 2e^x - 2xe^x \, dx = 2\int_0^1 (1-x)e^x \, dx.$$

This integral is perfect for integration by parts, since it is a product of $(1-x)$ and e^x, and if we take $u = 1 - x$ and $dv = e^x \, dx$, then $du = -dx$ is simpler than u but $v = e^x$ is no more complicated than dv. Using the integration by parts formula $\int u \, dv = uv - \int v \, du$ we have

$$\begin{aligned} 2\int_0^1 (1-x)e^x \, dx &= 2\left((1-x)e^x\Big]_0^1 - \int_0^1 e^x \cdot (-dx) \right) \\ &= 2\left((1-x)e^x\Big]_0^1 + e^x\Big]_0^1 \right) \\ &= 2\left((0-1) + (e-1)\right) \\ &= 2e - 4 \end{aligned}$$

6. [17 points] Find the volume of the solid obtained by rotating the region bounded by $x = y^2$ and $y = x$ around the line $x = 1$.



Since the rotation is about a vertical axis, slices of the solid should be horizontal. Thus we want to use y as our variable.

A typical slice will be a washer, and the outer radius will be the distance from the line $x = 1$ to the curve $x = y^2$, which is given by $1 - y^2$. The inner radius will be the distance from the line $x = 1$ to the curve $x = y$, which is given by $1 - y$. So the volume of this typical slice will be

$$\pi((1 - y^2)^2 - (1 - y)^2)\Delta y.$$

The volume of the solid is the integral of all these slice volumes from the lowest slice ($y = 0$) to the highest slice ($y = 1$), which is

$$\int_0^1 \pi((1 - y^2)^2 - (1 - y)^2)\, dy.$$

We compute this integral as follows:

$$\begin{aligned}
\int_0^1 \pi((1 - y^2)^2 - (1 - y)^2)\, dy &= \pi \int_0^1 1 - 2y^2 + y^4 - (1 - 2y + y^2)\, dy \\
&= \pi \int_0^1 2y - 3y^2 + y^4\, dy \\
&= \pi \left(y^2 - y^3 + \frac{y^4}{4} \Big]_0^1 \right) \\
&= \frac{\pi}{4}
\end{aligned}$$