College of the Holy Cross | Prof. Jones
Math 126, Spring 2009 | February 18, 2009

Exam 1

Name:

You must explain all your work to receive credit for your answers

These problems are not necessarily arranged in ascending order of difficulty. Work them in an order that will maximize your score. If you need more space, use the back of the page. *Good luck!*

Problem	Score	Problem	Score
1		4cd	
2			
3			
4ab		Total	

1. Suppose that a car's velocity is given by $v(t) = \sqrt{1+t^2}$, where t is measured in seconds and v is measured in miles per hour.

   (a) [5 points] What does the quantity $\int_0^4 v(t)\,dt$ represent in physical terms?

   The velocity is the rate of change of the position of the car, or in equations, $v(t) = s'(t)$. Thus $\int_0^4 v(t)\,dt$ is the integral from 0 to 4 of the rate of change of the distance. By the net change theorem from class, this means that $\int_0^4 v(t)\,dt$ is the change in distance from $t = 0$ to $t = 4$, or in other words the distance traveled by the car in that period.

   (b) [10 points] Using the midpoint rule, approximate the total distance traveled by the car in the first eight seconds. Use 4 subintervals.

   The subintervals are $[0, 2], [2, 4], [4, 6]$, and $[6, 8]$. Their midpoints are $1, 3, 5$, and 7.

   The midpoint rule says that the total area is approximately

   $$2f(1) + 2f(3) + 2f(5) + 2f(7),$$

   which equals

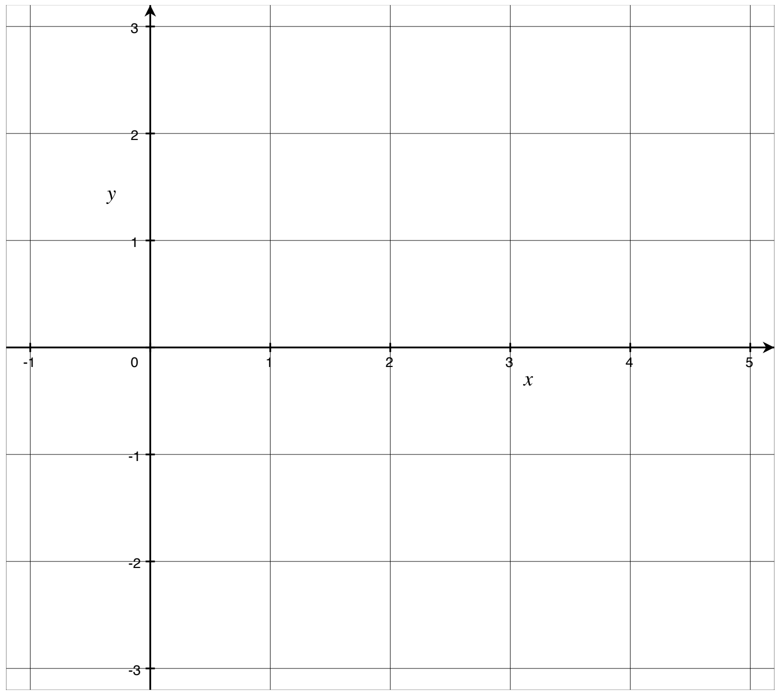
   $$2(\sqrt{1+1^2} + \sqrt{1+3^2} + \sqrt{1+5^2} + \sqrt{1+7^2}),$$

   which equals

   $$2(\sqrt{2} + \sqrt{10} + \sqrt{26} + \sqrt{50}),$$

   and this comes out to about 33.49.

2. Suppose that $f(t)$ has the following graph (Note: the graph was drawn in by hand. See your exam to remember how it looked):



Consider the function $g(x) = \int_0^x f(t)\,dt$. *Note that this is not the same function as the one graphed above.*

(a) [6 points] Find $g(0), g(1), g(2), g(3), g(4)$, and $g(5)$. $g(0) = \int_0^0 f(t)\,dt$, which is the area under the above graph from $t = 0$ to $t = 0$, which is just 0.

$g(1) = \int_0^1 f(t)\,dt$, which is the area under the above graph from $t = 0$ to $t = 1$, which is $3/2$ (using the area formula for a triangle).

$g(2) = 3$

$g(3) = 7/2 = 2.5$

$g(4) = 2$

$g(5) = 2.5$

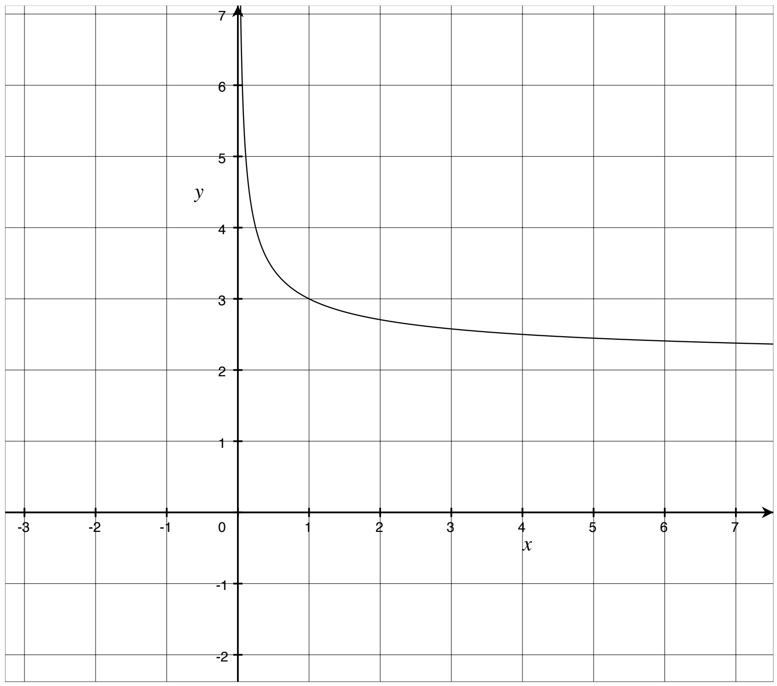
(b) [8 points] Find all local maxima and local minima for $g(x)$ with x between 0 and 5. At what x-value in $[0, 6]$ does the absolute max occur? Note: there was a typo on the $[0, 6]$; should have been $[0, 5]$.

A local max happens when $g(x)$ changes from increasing to decreasing. The only place this happens is $x = 2$ (we go from adding positive area to adding negative area) so there is a local max there and it's the only one. (Note: if you said that $x = 5$ was also a local max, that's correct in spirit but not correct technically, since a local max can only occur when the function is defined at least a little on either side of the point in question. No points were deducted for putting $x = 5$.)

A local min happens when $g(x)$ changes from decreasing to increasing. The only place this happens is $x = 4$ so there is a local min there and it's the only one. (No points were deducted if you put $x = 0$ as well).

The absolute max also occurs at $x = 2$, since there are no other local maxes and $g(2)$ is greater than $g(0)$ and $g(6)$.

3. [15 points] Find the area under the curve $y = 2 + \frac{1}{\sqrt[4]{x}}$ (pictured) from $x = 1$ to $x = 4$.



This problem says to find the area, not an approximation of the area. So we need to evaluate a definite integral rather than employing one of our approximations like the Midpoint Rule or the Trapezoid Rule.

The area in question is

$$\int_1^4 2 + \frac{1}{\sqrt[4]{x}}\, dx$$

which simplifies to

$$\int_1^4 2 + x^{-1/4}\, dx$$

An antiderivative of $2 + x^{-1/4}$ is $2x + \frac{4}{3}x^{3/4}$, and so we get

$$2x + \frac{4}{3}x^{3/4}\Big]_1^4 = 8 + \frac{4}{3}4^{3/4} - \left(2 + \frac{4}{3}\right) \approx 11.77 - 10/3 \approx 8.44$$

4. Find the following integrals:

(a) [14 points] $\int_1^e x^2 \ln x \, dx$

No substitution seems to work, and expression to be integrated is a product, so we use integration by parts. Selecting $u = \ln x$ seems like a good move, since $du = \frac{1}{x}\, dx$ and this will combine nicely with v. We have $dv = x^2 \, dx$, so $v = \frac{1}{3}x^3$. Using the integration by parts formula $\int u \, dv = uv - \int v \, du$ gives us

$$\int_1^e x^2 \ln x \, dx = (\ln x)\frac{1}{3}x^3\Big]_1^e - \int_1^e \frac{1}{3}x^3 \frac{1}{x} \, dx$$

This last integral is

$$\frac{1}{3}\int_1^e x^2 \, dx$$

which equals

$$\frac{1}{9}x^3\Big]_1^e$$

So we must compute

$$(\ln x)\frac{1}{3}x^3\Big]_1^e - \frac{1}{9}x^3\Big]_1^e,$$

and this is $(1/3)e^3 - ((1/9)e^3 - (1/9))$, which equals $(2/9)e^3 + (1/9)$.

(b) [14 points] $\int_0^1 te^{t^2} \, dt$

Here the substitution $u = t^2$ works well, since $du = 2t \, dt$, and this is nearly present in the integrand. Solving for $t \, dt$ gives $du/2 = t \, dt$. Convert the limits of integration from t to u: when $t = 0$, $u = 0^2 = 0$ and when $t = 1$, $u = 1^2 = 1$ (so in this case the limits don't change). Our integral is now

$$\frac{1}{2}\int_0^1 e^u \, du$$

which is

$$\frac{1}{2}\left(e^u]_0^1\right) = \frac{1}{2}(e - 1).$$

(c) [14 points] $\int x \cos x \, dx$

Use integration by parts. Let $u = x$, since du will be simpler (this is not the case if we pick $u = \cos x$). So $dv = \cos x \, dx$. We now have $du = dx$ and $v = \sin x$. So our integral becomes

$$x \sin x - \int \sin x \, dx$$

which is

$$x \sin x - (-\cos x) + C$$

and this is

$$x \sin x + \cos x + C.$$

(d) [14 points] $\int \frac{x}{2x^4 + 6x^2}\, dx$. (Hint: begin with a substitution.)

Begin with the substitution $u = x^2$. Then $du = 2x\, dx$, so $x\, dx = du/2$. The integral becomes

$$\frac{1}{2} \int \frac{1}{2u^2 + 6u}\, du$$

Factoring out a 1/2 gives

$$\frac{1}{4} \int \frac{1}{u^2 + 3u}\, du.$$

We now use partial fractions:

$$\frac{1}{u(u+3)} = \frac{A}{u} + \frac{B}{u+3}$$

Multiplying through by $u(u+3)$ gives $A(u+3) + Bu = 1$. Setting $u = -3$ and then $u = 0$ gives $A = 1/3$ and $B = -1/3$. Thus our integral is

$$\frac{1}{4} \int \frac{1/3}{u} - \frac{1/3}{u+3}\, du.$$

And this is

$$\frac{1}{12} \int \frac{1}{u}\, du - \frac{1}{12} \int \frac{1}{u+3}\, du.$$

Making the substitution $w = u+3$ in the second integral and then substituting back to x leaves us with the answer

$$\frac{1}{12} \left(\ln |x^2| - \ln |x^2 + 3| \right) + C.$$