Math 126: how to decide what integration technique to use

Here is a list that will help you decide what techniques to use to find an integral. These apply to both definite and indefinite integrals.

1. Simplify. Sometimes you can spot a simplification that allows you to do the integral. Examples are

$$\int \frac{3-2x}{x^2}\,dx \qquad \int (\sqrt{x}-2)^2\,dx$$

   For the first integral, you can break the fraction into $\frac{3}{x^2} - \frac{2}{x}$ (note that this would *not* be possible if the subtraction was in the denominator) and for the second integral, you can FOIL the integrand into something that you can integrate.

2. Aside from simplifying, the first two things to scan for are the Big Two techniques: u-substitution and integration by parts. These are the most common and most versatile techniques, and you should make yourself very familiar with them.

   - Substitution is the most common technique. It's optimally used when the integral contains a composition of functions (an inside function and an outside function, like $\sin(x^2)$) *and* there's a term present that serves as du. Some integrals that are ripe for substitution are

$$\int_{-3}^{2} x^2 e^{x^3}\,dx \qquad \int \cos x \sin(\sin x)\,dx \qquad \int \frac{\ln x}{x}\,dx$$

     In the first integral, take $u = x^3$, so that $du/3 = x^2\,dx$. In the second, take $u = \sin x$, and in the third, take $u = \ln x$ (then $du = 1/x\,dx$ is right there in the the integrand).

     Remember that you must convert the whole integral to u's, including the dx. Sometimes this can mean solving for dx, as in

$$\int \frac{1}{3-7x}\,dx$$

     Here, let $u = 3-7x$, meaning $du = -7\,dx$. To make the new integral all in terms of u, note that $dx = (-1/7)du$, and plug this in for dx.

     Sometimes you need to make a substitution just to set up the integral to have some other technique applied to it. Examples are

$$\int e^{\sqrt{x}}\,dx \qquad \int \frac{x}{1-x}\,dx$$

     For the first integral, the substitution $u = \sqrt{x}$ (which means $du = 1/(2\sqrt{x})$, which is in turn $1/(2u)$ sets up a use of integration by parts. For the second, setting $u = 1-x$ means $x = 1-u$ and leads to the integral

$$\int \frac{1-u}{u} \cdot (-du) = -\int \frac{1}{u} - 1\,du.$$

- Integration by parts is built to work on integrals that involve the product of functions, especially when taking the derivative of one of the functions in the product makes it simpler. Integrals ripe for integration by parts are

$$\int xe^x \, dx \qquad \int x^2 \ln x \, dx$$

  The formula for integration by pats is $\int u \, dv = uv - \int v \, du$. In the first example above, we would take $u = x$ (since its derivative gets much simpler) and $dv = e^x \, dx$. In the second example, take $u = \ln x$ and $dv = x^2 \, dx$. Note that u and dv must take up the *whole* integrand.

  Sometimes integration by parts needs to be applied multiple times, as with the integrals

$$\int x^2 e^x \, dx \qquad \int e^{2x} \sin x \, dx$$

3. A last thing to be on the lookout for is an integral that is just right for one of our specialized techniques – partial fractions, trig substitution, or a trigonometric integral?

- Partial fractions works on integrals that are ratios of two polynomials, usually where the denominator can be factored. Some examples are:

$$\int \frac{x+1}{x^2-4} \, dx \qquad \int_2^4 \frac{1}{x^3+x^2} \, dx \qquad \int \frac{x^2-x}{x^3+5x^2+6x} \, dx$$

- Trig substitution works on integrals that involve square roots of sums or differences, namely those of the form $\sqrt{x^2-a^2}$, $\sqrt{x^2+a^2}$, and $\sqrt{a^2-x^2}$. Some examples are

$$\int \frac{\sqrt{9-x^2}}{x^2} \, dx \qquad \int \frac{1}{x^2\sqrt{x^2+4}} \, dx$$

- Trigonometric integrals, in this course, have the form of a power of sine times a power of cosine. To do them, use the identity $\sin^2 x + \cos^2 x = 1$ to convert all but one of either the sines or cosines to the other, and then make a substitution. Examples are

$$\int \cos^3 x \sin^2 x \, dx \qquad \int \cos^{15} x \sin^3 x \, dx$$

  In the first integral, use $\cos^2 x = 1 - \sin^2 x$ to convert the integrand to $\cos x(1 - \sin^2 x) \sin^2 x$, and then make the substitution $u = \sin x$ (note that $du = \cos x$ is right there in the integrand).