Math 126: Examples with the Fundamental Theorem of Calculus

The Fundamental Theorem says:

1. Let $g(x) = \int_a^x f(t)dt$ for any number a. Then $g'(x) = f(x)$.

2. The Evaluation Theorem: $\int_a^b f(x)dx = F(b) - F(a)$, where $F'(x) = f(x)$.

(Notes: Above is the order that the book has the two parts in; in class, I put them in the other order. Here I'll conform to the book's order, since that's what it refers to in the directions for the homework. Also, the number a can be any real number – for instance, 0, 10, or π. In class, I only did the case $a = 0$.)

Part (1) says, loosely, that if we take the derivative of the integral of the function f, then we just get f back. So differentiation and integration are inverse operations. Part (2) says that we can find areas easily as long as we know an antiderivative of f.

As an example of how we can use the fundamental theorem, consider the following function:

$$g(x) = \int_0^x t + e^t dt.$$

This function measures the area under the curve $y = t + e^t$ from 0 to x. Let's find $g'(x)$ two ways: using part (1) of the Fundamental Theorem, and using part (2).

Using part (1) is very quick: we just take the function inside the integral, namely $f(t) = t + e^t$, and plug in x for t. By part (1), this is the derivative for $g(x)$. So $g'(x) = x + e^x$.

We can also use part (2) to actually find $g(x)$, and then take its derivative using our knowledge from Math 125. An antiderivative for $t + e^t$ is $\frac{1}{2}t^2 + e^t$, so we have

$$g(x) = \int_0^x t + e^t dt = \frac{1}{2}t^2 + e^t \Big]_0^x = \frac{1}{2}x^2 + e^x - (0 + e^0) = \frac{1}{2}x^2 + e^x - 1.$$

Then $g'(x) = \frac{1}{2} \cdot 2x + e^x + 0 = x + e^x$.