College of the Holy Cross
Math 211, Fall 2008

Prof. Jones
October 2, 2008

Exam 1 Solutions

Name:

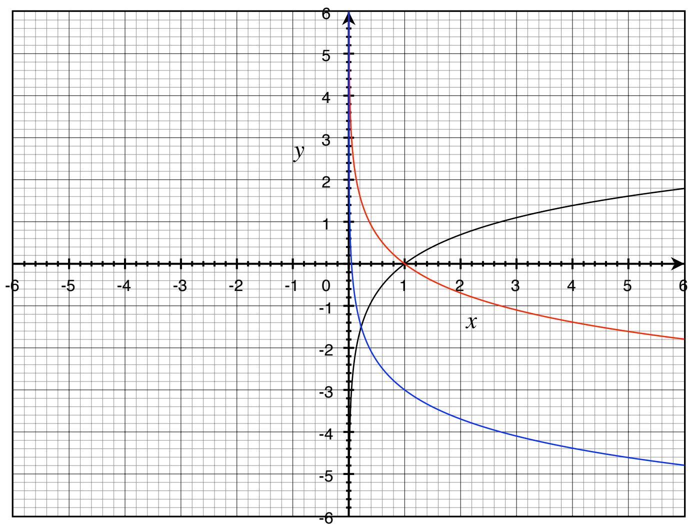
You must explain all your work to receive credit for your answers

These problems are not necessarily arranged in ascending order of difficulty. Work them in an order that will maximize your score. If you need more space, use the back of the page. *Good luck!*

Problem	Score	Problem	Score
1		5	
2		6	
3		7	
4		Total	

1. [10 points] Sketch the graph of the function $f(x) = -3 - \ln x$. Label at least one point in your graph. If you draw multiple functions on one graph as part of your work, express your final answer as a new graph showing only the function $f(x) = -3 - \ln x$.

   Begin with the graph of $y = \ln x$, shown in the graph below as the black line. Then obtain the graph of $y = -\ln x$ by reflecting about the x-axis. The result is shown as the red line in the graph below. Finally, obtain the graph of $y = -3 - \ln x$ by shifting down three units; the result is shown as the blue line on the graph. Note that shifting and then reflecting here gives the wrong graph, namely $y = -(3+\ln x)$.



   The graph of $y = -3 - \ln x$ goes through the point $(1, -3)$.

2. [10 points] The table below shows the entries for a linear function, an exponential function, and a trigonometric function:

x	$f(x)$	$g(x)$	$h(x)$
-2	24	1	16
-1	12	0	10
0	6	-1	4
1	3	0	-2
2	1.5	1	-8
3	0.75	0	-14

(a) Match each of the functions f, g, and h with its correct type (linear, exponential, or trigonometric). **Explain your answers completely.**

$f(x)$ is exponential, since for each increase in x, the value $f(x)$ is multiplied by the same number, namely 1/2.

$g(x)$ is trigonometric, since it oscillates among the vavlues $1, 0$, and -1.

$h(x)$ is linear, since for each increase in x the value $h(x)$ diminishes by a constant amount, namely 6.

(b) Fill in the last row of the table.

Multiplying $f(2)$ by 1/2 gives 0.75, which is $f(3)$. Making $g(x)$ continue its oscillatory pattern means $g(3) = 0$. Subtracting 6 from $h(2)$ gives -14, which is $h(3)$.

3. A population of mold has begun to develop on a piece of bread you brought back to your dorm from Kimball. Initially, the population is only 100 cells. However, this is a lively strain of mold, and after one hour it has tripled. Assume the cells continue to grow exponentially at this rate. Use the variables t for time and P for the number of mold cells.

(a) [4 points] How many mold cells are on the bread after four hours?

When $t = 0$, the population is 100

When $t = 1$, the population is $100 \cdot 3 = 300$

When $t = 2$, the population is $300 \cdot 3 = 900$

When $t = 3$, the population is $900 \cdot 3 = 2700$

When $t = 4$, the population is $2700 \cdot 3 = 8100$

(b) [5 points] How many mold cells are on the bread after t hours?

With each increase in t by one, the mold population triples, and so is multiplied by 3. Thus there will be a 3^t involved in our function. Since the initial mold population is 100, the function is

$$P(t) = 100 \cdot 3^t.$$

(c) [5 points] Public safety will confiscate the bread if the number of mold cells reaches one hundred thousand. First say during which hour this will happen (you may want to use your calculator). Then give the *exact* answer (don't use your calculator).

When $t = 6$, the mold population is $100 \cdot 3^6$, which is 72900. When $t = 7$, the mold population is $100 \cdot 3^7$, which is 218700. So the population exceeds 100000 sometime during the 6th hour.

The exact time can be found by solving the equation $100000 = 100 \cdot 3^t$, which is the same as

$$1000 = 3^t.$$

Taking $\log_3$ of both sides gives $t = \log_3 1000$. Alternatively, you can take ln of both sides, which gives $\ln 1000 = \ln 3^t$. Using one of the log rules, the right-hand side is $t \ln 3$, and so we have $t = (\ln 1000)/(\ln 3)$. Indeed, this is the same as $t = \log_3 1000$.

4. Let $f(x) = \ln x$ and $g(x) = \frac{1}{x}$.

(a) [5 points] Find the domains of f and g.

$\ln x$ is not defined for $x = 0$ or when x is negative. So the domain of f is all $x > 0$, or $(0, \infty)$.

$\frac{1}{x}$ only fails to be defined when $x = 0$, and so the domain of g is all $x \neq 0$, or $(-\infty, 0) \cup (0, \infty)$.

(b) [8 points] Find $(f \circ g)(x)$, and find its domain.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \ln \frac{1}{x}.$$

In order for this expression to be defined, the expression we plug into ln must be greater than zero. Thus we need

$$\frac{1}{x} > 0,$$

and this happens precisely when $x > 0$. Thus the domain of $(f \circ g)$ is all $x > 0$, or $(0, \infty)$.

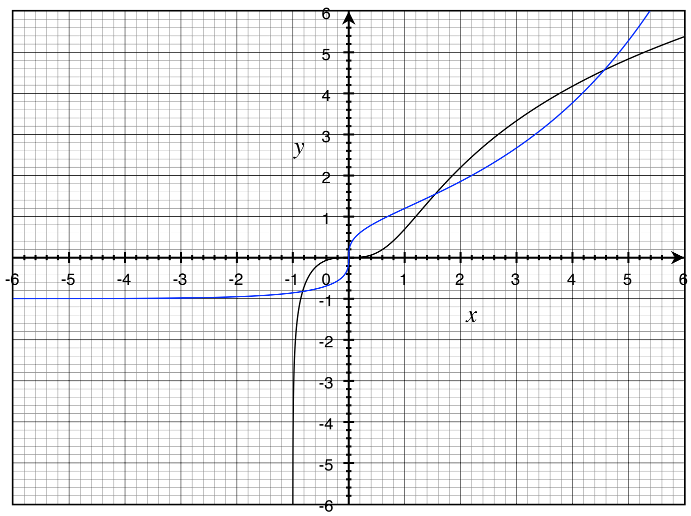
(c) [8 points] Find $(g \circ f)(x)$, and find its domain.

$$(g \circ f)(x) = g(f(x)) = g(\ln x) = \frac{1}{\ln x}.$$

For this to be defined, we need first to ensure that $\ln x$ is defined, so $x > 0$. However, we also need to ensure that the denominator is not 0, in other words $\ln x \neq 0$. But $\ln x = 0$ precisely when $x = 1$ (this is something you should know; you can also derive it by exponentiating both sides of $\ln x = 0$). So the domain is all $x > 0$ except $x = 1$, or $(0, 1) \cup (1, \infty)$.

5. (a) [7 points] Suppose that f is given by the graph below. Sketch the graph of f^{-1}.

To get the graph of f^{-1}, reflect the graph of f about the line $y = x$. The result is shown in blue on the graph below, while the graph of f is shown in black.



(b) [8 points] Suppose $f(x) = \ln(x^3 + 1)$. Find $f^{-1}(x)$.

First write $y = \ln(x^3 + 1)$. Now we solve for x:

$$e^y = e^{\ln(x^3+1)}$$
$$e^y = x^3 + 1$$
$$e^y - 1 = x^3$$
$$\sqrt[3]{e^y - 1} = x$$

We now swap x and y, and wind up with

$$y = \sqrt[3]{e^x - 1}$$

and so

$$f^{-1}(x) = \sqrt[3]{e^x - 1}.$$

6. Simplify the following expressions as much as possible:

   (a) [5 points] $\ln(e^4 \cdot e^7) + \log_5 \frac{1}{5}$

   These problems are both done almost entirely by using the log rules that we learned in section 1.6 (see p. 68 of the book).

$$\begin{aligned}
&= \ln(e^{11}) + \log_5 \tfrac{1}{5} && \text{since } a^x \cdot a^y = a^{x+y} \\
&= \ln(e^{11}) + \log_5 5^{-1} && \text{since } 1/a = a^{-1} \\
&= 11 + (-1) && \text{since } \log_a a^x = x\text{, so in particular } \ln e^x = x \\
&= 10
\end{aligned}$$

   (b) [5 points] $\ln 32 + \ln \frac{1}{2} - 4 \ln 2$

$$\begin{aligned}
&= \ln 32 + \ln \tfrac{1}{2} - \ln 2^4 && \text{since } r \ln x = \ln x^r \\
&= \ln 16 - \ln 2^4 && \text{since } \ln x + \ln y = \ln xyb \\
&= \ln 16 - \ln 16 \\
&= 0
\end{aligned}$$

7. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 miles, and in June it cost her \$460 to drive 800 miles.

(a) [5 points] Express the monthly cost C as a function of the distance driven d, assuming that they are related linearly.

We want C as a function of d, so we should express our points as (d, C). Thus we know that $(480, 380)$ and $(800, 460)$ are points on our line. This means the line has slope

$$\frac{460-380}{800-480} = \frac{80}{320} = \frac{1}{4}.$$

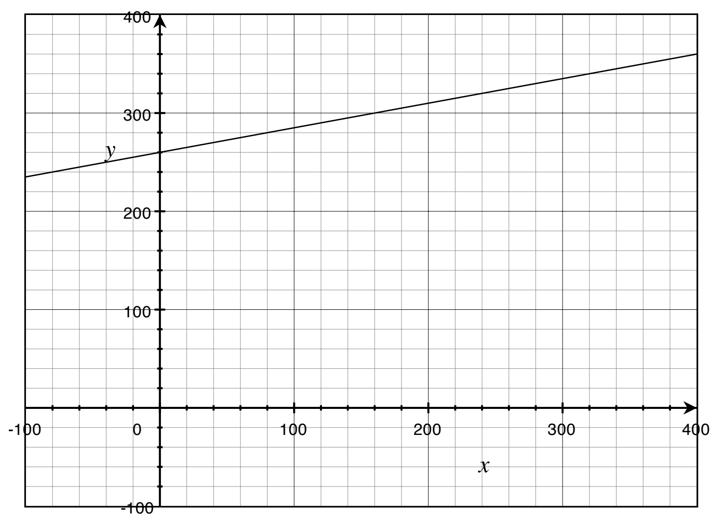
To find the equation for the line, we use point-slope form with the point $(800, 460)$:

$$C - 460 = \frac{1}{4}(d - 800)$$

$$C - 460 = \frac{1}{4}d - 200$$

$$C = \frac{1}{4}d + 260$$

(b) [5 points] Sketch a graph of the linear function. What does the slope represent?



The slope represents the fact that for every additional mile Lynn drives, her monthly cost increases by 25 cents.

(c) [5 points] What will be the cost of driving 1500 miles?

We plug $d = 1500$ in to our equation:

$$C = \frac{1}{4}(1500) + 260$$

$$C = 375 + 260$$

$$C = 635$$

(d) [5 points] Find the exact coordinates of the point on the graph that has C-coordinate 1000. What information does this point tell you?

When $C = 1000$, it means d is given by the equation

$$1000 = \frac{1}{4}d + 260$$

Solving this for d gives

$$740 = \frac{1}{4}d$$

$$2960 = d$$

So the point is $(2960, 1000)$. This means that in order for the cost to be \$1000, Lynn must drive 2960 miles.

8. *Bonus Problem* [*5 points*]: Let

$$f(x) = \begin{cases} 1 + x & \text{if } x > 0 \\ \dfrac{\sin x}{x} & \text{if } x < 0 \end{cases}$$

Find $\lim_{x \to 0} f(x)$. Carefully justify your answer. Hint: we discussed the behavior of $\frac{\sin x}{x}$ near $x = 0$ in class on Tuesday.

We saw in class that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. That means that as we approach $x = 0$ from either side, the values of $\frac{\sin x}{x}$ approach 1. In particular, this holds if we come in only from the left, or in other words,

$$\lim_{x \to 0^-} \frac{\sin x}{x} = 1.$$

And therefore $\lim_{x \to 0^-} f(x) = 1$, since for $x < 0$ we have $f(x) = \frac{\sin x}{x}$.

On the other hand, it's also true that $\lim_{x \to 0} 1 + x = 1$, as is shown by the limit laws from Section 2.3. That means that as we approach $x = 0$ from either side, the values of $1 + x$ approach 1. In particular, this holds if we come in only from the right, or in other words,

$$\lim_{x \to 0^+} 1 + x = 1.$$

And therefore $\lim_{x \to 0^+} f(x) = 1$, since for $x > 0$ we have $f(x) = 1 + x$.

Since the one-sided limits from both sides agree, we have

$$\lim_{x \to 0} f(x) = 1.$$