Research Statement – Rafe Jones

My primary research interest is iteration of polynomials and rational functions over finite and p-adic fields. In studying these questions, I use tools from algebraic number theory, finite group theory, and probablility theory. In my dissertation I determine the density of the "hyperbolic Mandelbrot set" of $\overline{\mathbb{F}}_p$, that is

$$J_p = \{c \in \overline{\mathbb{F}}_p : 0 \text{ is purely periodic under iteration of } x^2 + c\}. \tag{1}$$

My main result is that

$$\lim_{k\to\infty} \frac{\# J_p \cap \mathbb{F}_{p^k}}{p^k} = 0. \tag{2}$$

This result has an application to the hyperbolic subset of the p-adic Mandelbrot set, whose complex analogue has been much studied. Much of the proof of (2) is an analysis of the Galois tower formed by the splitting fields of iterates of $y^2 + x \in \mathbb{F}_p(x)[y]$. Similar towers have been studied recently by Odoni [3], Aitken et al [1] and others. I introduce a stochastic process associated to any tower of Galois extensions, and show that the process associated to the tower mentioned above is a martingale. A martingale convergence theorem is then instrumental in proving (2). This method of proof appears to be highly unusual, and may well have applications to other density questions in number theory.

This work offers several avenues for further research. For instance, one can consider other algebraic families of functions in $\overline{\mathbb{F}}_p(x)$. It would be nice to know if there exists such a family with a fixed critical point α such that

$$\{c \in \mathbb{F}_{p^k} : \alpha \text{ is purely periodic under iteration of } x^2 + c\}$$

has positive density as $k \to \infty$.

Another direction for further research follows from considering $x^2 + c$ for a fixed c. For instance, if we take $c \in \mathbb{F}_p$, we can ask about the density as $k \to \infty$ of the set

$$\{\beta \in \mathbb{F}_{p^k} : \beta \text{ is purely periodic under } x^2 + c\}.$$

Using the Tchebotarev Theorem for function fields, this boils down to a question about Galois groups similar to those considered in my thesis.

The Galois-theoretic portion of my thesis appears to apply to fields other than $K = \mathbb{F}_p(x)$. If we put $K = \mathbb{Q}(x)$ then we can consider, for $a \in \mathbb{Z}$, specializations f_a of $f_x = y^2 + x \in \mathbb{Q}(x)[y]$. Values of a for which the Galois group of the nth iterate of f_a over $\mathbb{Q}$ is smaller than the corresponding group for f_x over $\mathbb{Q}(x)$ correspond to integral points on a certain hyperelliptic curve. It may thus be possible to use techniques from algebraic geometry to get further results. Moreover, my results imply that for most $a \in \mathbb{Z}$ a certain infinite set of elements exists in the Galois tower formed by all iterates of f_a. This builds on work of Odoni [3] and Stoll [4].

More generally, I plan to learn more about the theory of tree representations of Galois groups, which Aitken et al [1] and Boston [2] have proposed as a vehicle for studying the maximal extension of a number field unramified outside a finite set of primes. Such representations arise naturally from iterates of polynomials over number fields. I also plan to deepen my knowledge of group theory and probability theory with an eye to applying them to distribution problems in number theory and algebraic geometry. Finally, I plan to find applications of my results to the dynamics of more general p-adic rational functions.

References

[1] Wayne Aitken, Farshid Hajir, and Christian Maire. Finitely ramified iterated extensions. Available at http://arxiv.org/abs/math.NT/0408170.

[2] Nigel Boston. Tree representations of galois groups. Available at http://www.math.uiuc.edu/~boston/treereps.pdf.

[3] R. W. K. Odoni. Realising wreath products of cyclic groups as Galois groups. *Mathematika*, 35(1):101–113, 1988.

[4] Michael Stoll. Galois groups over **Q** of some iterated polynomials. *Arch. Math. (Basel)*, 59(3):239–244, 1992.