# CO-CIRCULAR RELATIVE EQUILIBRIA OF FOUR VORTICES

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Abstract. We study the co-circular relative equilibria (planar central configurations) in the four-vortex problem using methods suggested by the study of co-circular central configurations in the Newtonian four-body problem in recent work of Cors and Roberts. Using mutual distance coordinates, we show that the set of four-vortex relative equilibria is a two-dimensional surface with boundary curves representing kite configurations, isosceles trapezoids, and degenerate configurations with one zero vorticity. We also show that there is a constraint on the signs of the vorticities in these configurations; either three or four of the vorticities must have the same sign, in contrast to the non-co-circular cases studied by Hampton, Roberts, and Santoprete.

## 1. Introduction

Understanding central configurations is a problem of fundamental importance in celestial mechanics (for instance, see [11]). Recent years have seen heightened interest in the study of central configurations, in part due to the fact that advances in computing power have made it possible to utilize tools from algebraic geometry to study such problems. These tools have led to breakthroughs such as the proof that there are only finitely many central configurations for each collection of positive masses in the four body problem ([6]), and the proof of finiteness in generic cases of the five-body problem ([5, 1]).

Similarly useful is the study of relative equilibrium configurations of collections of *Helmholtz vortices*, [7, 11]. Helmholtz vortices, thought of as whirlpools lying in an infinite plane composed of a perfect fluid, were first introduced as a means of modeling the interactions of two-dimensional slices of collections of columnar vortex filaments. The study of relative equilibria of vortices has applications that range from basic fluid mechanics to the study of how cyclones and hurricanes interact and evolve over time.

Algebraically, the equations defining relative equilibria of vortices are very similar to those defining relative equilibria of masses. Suppose vortices of strengths $\Gamma_i$ (unlike the masses in the Newtonian problem, these can have positive or negative real values) are initially located at positions $q_i \in \mathbb{R}^2$. Writing $r_{ij} = \|q_i - q_j\|$ for the mutual distance, then we have a relative equilibrium if for all $i$,

$$\sum_{j\neq i} \Gamma_j \frac{(q_i - q_j)}{r_{ij}^2} = -\lambda(q_i - c), \tag{1.1}$$

where $\lambda$ is a constant and $c$ is the center of rotation. The equations (1.1) differ from their Newtonian equivalents because of the $r_{ij}^2$ in the denominators (where

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$r_{ij}^3$ appears in the equations for relative equilibria of masses). The difference is caused by a logarithmic potential in the vortex case that replaces the gravitational potential in the Newtonian case.

In this paper, we study relative equilibria of collections of four point vortices whose locations lie on a circle in the plane (the co-circular configurations in the title). The inspiration for this study can be found in a recent paper in which Cors and Roberts study the corresponding problem for four co-circular masses under Newtonian gravity, [2]. Other articles devoted to the study of co-circular central configurations include [4, 9]. We also use a number of general results on the vortex problem from a second recent article by Hampton, Roberts, and Santoprete, [8]. We first present a set of equations in mutual distance coordinates whose solutions correspond to these configurations in §2. By analyzing the set of solutions of these equations, in §3 we obtain a surface in $\mathbb{R}^3$ whose points parametrize the family of co-circular relative equilibria. Next, in §4, we prove a result concerning the possible signs of the vorticities for a co-circular relative equilibrium. We discuss some constraints on the positions $q_i$ and the vorticities $\Gamma_i$ in relative equilibria in §5. Finally, we follow [2], *mutatis mutandis*, and analyze two symmetric cases (kites and isosceles trapezoids) in §§6 and 7. These cases correspond to boundary points of our surface.

## 2. Equations for relative equilibria in mutual distance coordinates

By using results from [8] on the general four-vortex problem and adapting results from [2] on the co-circular case of the four-body problem, in this section we will derive a set of equations characterizing the co-circular relative equilibria in the four-vortex problem.

By Equation (10) of [8], the following relation (a consequence of the Dziobek relations in the vortex case) is necessary and sufficient for the existence of a four-vortex relative equilibrium with mutual distances $r_{ij} > 0$, $1 \le i < j \le 4$:

$$(r_{13}^2 - r_{12}^2)(r_{23}^2 - r_{34}^2)(r_{24}^2 - r_{14}^2) - (r_{12}^2 - r_{14}^2)(r_{24}^2 - r_{34}^2)(r_{13}^2 - r_{23}^2) = 0. \tag{2.1}$$

For future reference, we note that this equation can be rearranged algebraically in many different ways. We will also need the following forms:

$$(r_{14}^2 - r_{24}^2)(r_{13}^2 - r_{34}^2)(r_{12}^2 - r_{23}^2) - (r_{14}^2 - r_{34}^2)(r_{13}^2 - r_{23}^2)(r_{12}^2 - r_{24}^2) = 0, \tag{2.2}$$

$$(r_{23}^2 - r_{24}^2)(r_{14}^2 - r_{34}^2)(r_{12}^2 - r_{13}^2) - (r_{24}^2 - r_{34}^2)(r_{13}^2 - r_{14}^2)(r_{12}^2 - r_{23}^2) = 0, \tag{2.3}$$

and

$$(r_{24}^2 - r_{23}^2)(r_{13}^2 - r_{34}^2)(r_{12}^2 - r_{14}^2) - (r_{34}^2 - r_{23}^2)(r_{13}^2 - r_{14}^2)(r_{12}^2 - r_{24}^2) = 0. \tag{2.4}$$

Now we impose the condition that the locations of the four vortices lie on a single circle in the plane. Numbering the positions sequentially around that circle, it follows that $r_{12}, r_{23}, r_{34}, r_{14}$ are the lengths of the exterior edges of a cyclic quadrilateral, and $r_{13}, r_{24}$ are the lengths of the diagonals. Letting

$$a = r_{12}r_{34} + r_{14}r_{23},\ b = r_{12}r_{14} + r_{23}r_{34},\ c = r_{12}r_{23} + r_{14}r_{34}, \tag{2.5}$$

from the Law of Cosines and the fact that opposite interior angles in the quadrilateral are supplementary, it follows that

$$r_{13}^2 = \frac{ab}{c} \tag{2.6}$$

$$r_{24}^2 = \frac{ac}{b}. \tag{2.7}$$

Multiplying the two equations above and taking square roots gives Ptolemy's theorem on cyclic quadrilaterals:

$$r_{13}r_{24} = r_{12}r_{34} + r_{14}r_{23}. \tag{2.8}$$

As in [2], we will always fix the numbering of the vortices so that $r_{12}$ is the largest exterior side length and we will normalize the unit of distance so $r_{12} = 1$. Then

$$r_{23}, r_{34}, r_{14} \leq 1. \tag{2.9}$$

As noted in [2], we also have

$$\frac{r_{13}}{r_{24}} = \frac{b}{c},$$

so

$$r_{13} - r_{24} \geq 0 \Leftrightarrow b - c \geq 0 \Leftrightarrow (r_{14} - r_{23})(r_{12} - r_{34}) \geq 0.$$

Since $r_{12} \geq r_{34}$ by our choice of labeling,

$$r_{14} \geq r_{23} \Leftrightarrow r_{13} \geq r_{24}. \tag{2.10}$$

We note some additional useful consequences of the equations above relating the diagonals of the cyclic quadrilateral to the exterior sides. In words, these inequalities will say that *the diagonals of the cyclic quadrilateral are longer than any exterior side on the opposite side of the diagonal from the longest exterior side.* For instance, from (2.6), notice that

$$r_{13}^2 - r_{14}^2 = r_{34}\left(\frac{r_{34}r_{23} + r_{23}^2 r_{14} + r_{14} - r_{14}^3}{r_{23} + r_{14}r_{34}}\right) > 0 \tag{2.11}$$

(since $r_{14} - r_{14}^3 \geq 0$ by (2.9)). By similar computations, we also have

$$r_{13}^2 - r_{34}^2 > 0, \tag{2.12}$$

$$r_{24}^2 - r_{23}^2 > 0, \tag{2.13}$$

$$r_{24}^2 - r_{34}^2 > 0. \tag{2.14}$$

Let $\Gamma_i \in \mathbb{R}\setminus\{0\}$, $i = 1, \dots 4$ denote the strengths (vorticities) of the four vortices. The derivation of (2.1) above and a computation analogous to that giving equations (16-18) in [2] leads to the following vorticity ratio formulas:

$$\frac{\Gamma_2}{\Gamma_1} = \frac{r_{23}r_{24}(r_{13}^2 - r_{14}^2)}{r_{13}r_{14}(r_{24}^2 - r_{23}^2)}, \tag{2.15}$$

$$\frac{\Gamma_3}{\Gamma_1} = \frac{r_{23}r_{34}(1 - r_{14}^2)}{r_{14}(r_{23}^2 - r_{34}^2)}, \tag{2.16}$$

$$\frac{\Gamma_4}{\Gamma_1} = \frac{r_{24}r_{34}(r_{13}^2 - 1)}{r_{13}(r_{24}^2 - r_{34}^2)}. \tag{2.17}$$

We can always normalize (choose units for vorticity) to set $\Gamma_1 = 1$. By (2.3), (2.11), (2.12), (2.13), and (2.14), the numerator in the formula for $\Gamma_2$ and the denominators in the formulas for $\Gamma_2$ and $\Gamma_4$ are always nonzero, so the values of $\Gamma_2$ and $\Gamma_4$ are

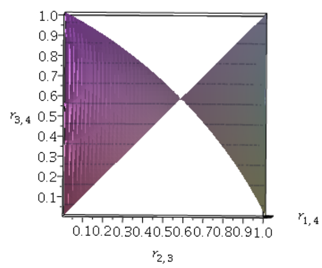


Figure 1: The surface $F_3(r_{23}, r_{34}, r_{14}) = 0$

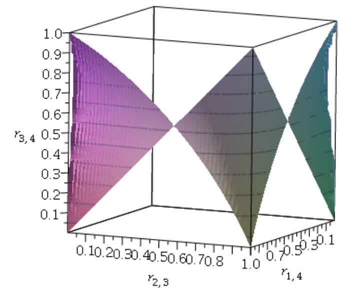


Figure 2: Another view

always determined by these. The equation (2.16) gives a well-defined value for $\Gamma_3$ unless $r_{23}^2 - r_{34}^2 = 0$. Looking at (2.4), (2.12), and (2.13), we see that this implies $1 - r_{14}^2 = 0$ so the quotient is actually indeterminate. If, on the other hand, the factor $1 - r_{14}^2$ vanishes, then (2.4) and (2.11) show that $r_{23}^2 - r_{34}^2 = 0$, or $1 - r_{24}^2 = 0$. When $r_{23}^2 - r_{34}^2 = 0$, an alternate formula for $\Gamma_3$ can be derived using (2.1):

$$\Gamma_3 = \frac{(r_{13}^2 - 1)(r_{24}^2 - 1)r_{23}^2}{(r_{24}^2 - r_{23}^2)(r_{13}^2 - r_{23}^2)}. \tag{2.18}$$

There are solutions with $r_{14} = r_{12} = r_{24} = 1$ corresponding to degenerate configurations with vortices $1, 2, 4$ forming an equilateral triangle and $\Gamma_3 = 0$. Similarly, there are degenerate configurations with $r_{13} = r_{12} = r_{23} = 1$ and $\Gamma_4 = 0$. The configurations with $r_{14} = r_{12} = 1$ and $r_{23} = r_{34}$ are the symmetric kites to be studied in §6.

Collecting all of the results stated above, we see the following statement.

**Theorem 2.1.** *A co-circular configuration of four vortices with mutual distances $r_{ij}$, vorticities $\Gamma_i$, and with $r_{12} = 1$, $r_{14} < 1$ and $\Gamma_1 = 1$ is a relative equilibrium if and only if the $r_{ij}$ and $\Gamma_i$ give a common zero of the following set of six polynomial equations:*

$$\begin{aligned}
F_1 &= r_{13}^2(r_{23} + r_{34}r_{14}) - (r_{34} + r_{14}r_{23})(r_{14} + r_{23}r_{34}) \\
F_2 &= r_{24}^2(r_{14} + r_{23}r_{34}) - (r_{34} + r_{14}r_{23})(r_{23} + r_{14}r_{34}) \\
F_3 &= (r_{13}^2 - 1)(r_{23}^2 - r_{34}^2)(r_{24}^2 - r_{14}^2) - (1 - r_{14}^2)(r_{24}^2 - r_{34}^2)(r_{13}^2 - r_{23}^2) \\
F_4 &= r_{13}r_{14}(r_{24}^2 - r_{23}^2)\Gamma_2 - r_{23}r_{24}(r_{13}^2 - r_{14}^2) \\
F_5 &= r_{14}(r_{23}^2 - r_{34}^2)\Gamma_3 - r_{23}r_{34}(1 - r_{14}^2) \\
F_6 &= r_{13}(r_{24}^2 - r_{34}^2)\Gamma_4 - r_{23}r_{24}(r_{13}^2 - 1).
\end{aligned} \tag{2.19}$$

*When $r_{12} = r_{14} = 1$, the equation $F_5 = 0$ is replaced by a similar equation $F_5' = 0$ derived from* (2.18).

## 3. The surface of co-circular relative equilibria

As suggested by the naive count of variables and equations in the system (2.19), with our normalizations, the set of co-circular relative equilibria is two-dimensional.

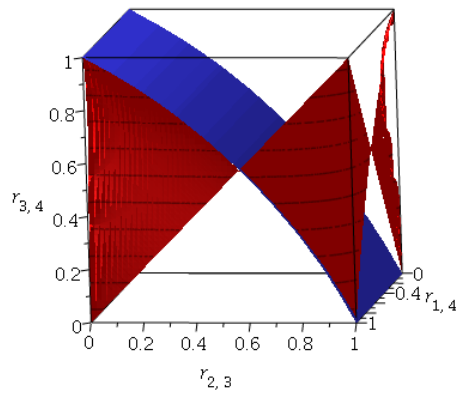


Figure 3: Plot of $r_{23}^2 + r_{23}r_{34} + r_{34}^2 - 1 = 0$ with the graph of $F_3(r_{23}, r_{34}, r_{14}) = 0$

The equations $F_4 = F_5 = F_6 = 0$ in Theorem 2.1 express the vorticities $\Gamma_2, \Gamma_3, \Gamma_4$ in terms of the $r_{ij}$. Moreover, we may use the equations $F_1 = 0$ and $F_2 = 0$ to write the squared diagonals $r_{13}^2$ and $r_{24}^2$ as functions of the other mutual distances as in Equations (2.6) and (2.7) above. Using these two relations one can think of $F_3$ as a function of the three exterior side lengths $r_{23}, r_{34}, r_{14}$:

$$F_3(r_{23}, r_{34}, r_{14}) = (r_{13}^2 - 1)(r_{23}^2 - r_{34}^2)(r_{24}^2 - r_{14}^2) - (1 - r_{14}^2)(r_{24}^2 - r_{34}^2)(r_{13}^2 - r_{23}^2),$$

and then the equation $F_3 = 0$ defines an algebraic surface in the $\mathbb{R}^3$ with coordinates $r_{23}, r_{34}, r_{14}$.

By (2.9), we can plot the set of points on which $F_3 = 0$ implicitly in the unit cube. Figure 1 shows the view of the surface looking along the positive $r_{14}$-axis toward the $r_{23}, r_{34}$-plane. There is a nearly-vertical portion of the surface that is obscured from this viewpoint, but visible in the rotated view in Figure 2. However, the entire implicit plot is symmetric across the plane $r_{14} = r_{23}$ (this can be seen by the fact that interchanging $r_{14}$ and $r_{23}$ takes (2.1) to (2.2)).

Therefore we can assume without loss of generality that $r_{14} \geq r_{23}$, and so also $r_{13} \geq r_{24}$ by (2.10). We will only consider that portion of the graph in the following. Because of the shape, we will refer to it as the *bowtie surface*.

We next consider what configurations correspond to points on the boundary curves. Note that if $r_{14} = r_{23}$, then equations (2.5), (2.6), and (2.7) imply that $r_{13} = r_{24}$ as well, so the only cases where $r_{14} = r_{23}$ are the configurations known as isosceles trapezoids. These will be studied in more detail in §7. We next note that since $1 = r_{12} \geq r_{14} \geq r_{23}$ the rest of the boundary is defined by $r_{14} = 1$. Substituting this into $F_3$ and factoring yields:

$$F_3(r_{23}, r_{34}, 1) = (r_{23} - r_{34})(r_{23} + r_{34})^2 \left(r_{23}^2 + r_{23}r_{34} + r_{34}^2 - 1\right).$$

The first factor vanishes on points corresponding to kite configurations where $r_{23} = r_{34}$. The kite cases will be completely characterized in §6.

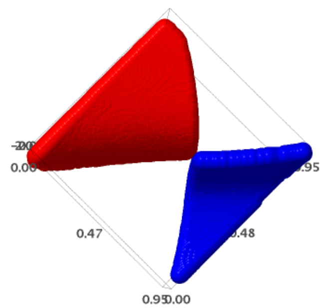


Figure 4: $\Gamma_3$, view along $r_{14}$ axis

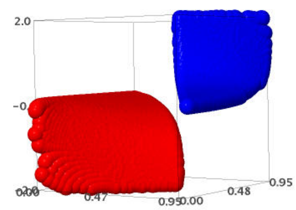


Figure 5: $\Gamma_3$, view from side

The second factor is never zero for positive mutual distances. Hence it is left to consider cases where

$$r_{23}^2 + r_{23}r_{34} + r_{34}^2 - 1 = 0.$$

Examining (2.7), we see that when $r_{12} = r_{14} = 1$, this equation is equivalent to $r_{24}^2 = 1$. Therefore, the vortices 1,2,4 are at the corners of an equilateral triangle and it follows by (2.18) that $\Gamma_3 = 0$. Thus, the points on this curved component of the boundary shown in Figure 3 correspond to degenerate configurations.

## 4. The signs of the vorticities

In this section we will analyze the possible signs of the $\Gamma_i$ in solutions of the system of equations from Theorem 2.1. We will see that in fact in any such relative equilibrium either all of the $\Gamma_i$ have the same sign, or else three of the $\Gamma_i$ have the same sign and the remaining vorticity has the opposite sign.

We were led to conjecture these patterns by plots showing the values for the vorticity $\Gamma_3$ obtained from the equation $F_5 = 0$ in (2.19) on the points of the bowtie surface defined by $F_3 = 0$. To generate the plots in Figures 4 and 5, we solved the equation $F_3 = 0$ numerically for $r_{14}$ as a function of $r_{23}$ and $r_{34}$ at a collection of points in the projection of the bowtie onto the $r_{23}, r_{34}$ plane, then plotted positive $\Gamma_3$ values in blue and negative $\Gamma_3$ values in red. Figure 4 shows a top view along the direction of the $r_{14}$-axis. Figure 5 shows the same plot of $\Gamma_3$-values, but from one side.

In the remainder of this section, we will give an analytic proof that $\Gamma_3$ takes opposite signs on the two lobes of the bowtie surface. We will need the following fact from [2]; this depends only on the geometry of the cyclic quadrilateral.

**Lemma 4.1** ([2], Lemma 4.6)**.** *Under the assumption $r_{14} \geq r_{23}$, and the consequence noted above in* (2.10)*, it follows that*

$$\frac{r_{13}}{r_{24}} \leq \frac{r_{14}}{r_{23}}.$$

*Proof.* For the convenience of the reader we reproduce the proof from [2]. From (2.6) and (2.7), and using the assumptions $r_{12} = 1$ and $r_{23} \leq r_{14}$, we have

$$\frac{r_{13}}{r_{24}} = \frac{b}{c} = \frac{r_{14} + r_{23}r_{34}}{r_{23} + r_{14}r_{34}} \leq \frac{r_{14}(1 + r_{34})}{r_{23}(1 + r_{34})},$$

which implies the claim. □

**Lemma 4.2.** *In all co-circular four-vortex relative equilibria as above* $\Gamma_2 > 0$.

*Proof.* From the equation $F_4 = 0$, we have

$$\Gamma_2 = \frac{r_{24}}{r_{13}} \cdot \frac{r_{23}}{r_{14}} \cdot \frac{(r_{13}^2 - r_{14}^2)}{(r_{24}^2 - r_{23}^2)}. \tag{4.1}$$

The inequality $\Gamma_2 > 0$ follows from (2.11) and (2.13). □

The portion of the bowtie surface with $r_{14} \geq r_{23}$ off the boundary curves is composed of two *lobes*: one (on the left in Figure 1) on which $r_{23} < r_{34}$, and a second on which $r_{23} > r_{34}$. We will call these *open subsets* of the bowtie surface lobe I and lobe II, respectively. The closures of the two lobes of the surface intersect only at the point corresponding to a degenerate configuration that is also a kite.

We will deal with the points in the interior of lobe II first, since they follow essentially the same patterns as those found by Cors and Roberts in the co-circular 4-body central configurations. We note that in [2] the inequality $r_{23} \geq r_{34}$ was deduced from the positivity of the masses $m_i$ (see §2.2 of [2]). However, this inequality holds by definition on our lobe II.

**Theorem 4.3.** *On lobe II, we have*

$$\Gamma_2 \geq \Gamma_4 \geq \Gamma_3 > 0.$$

*Hence all four of the vorticities have the same sign on lobe II.*

*Proof.* The inequality $\Gamma_2 \geq \Gamma_4$ follows from the equations $F_4 = 0$ and $F_6 = 0$, or from (2.15) and (2.17). These say

$$\begin{aligned} \Gamma_2 &= \frac{r_{23} r_{24} (r_{13}^2 - r_{14}^2)}{r_{13} r_{14} (r_{24}^2 - r_{23}^2)} \\ \Gamma_4 &= \frac{r_{34} r_{24} (r_{13}^2 - 1)}{r_{13} (r_{24}^2 - r_{34}^2)}. \end{aligned}$$

and the inequalities $r_{23} > r_{34}$, $r_{14} \leq 1$, and $r_{13} \geq r_{14}$ combine to give $\Gamma_2 \geq \Gamma_4$. Finally, $\Gamma_4 \geq \Gamma_3 > 0$ follows using Lemma 4.1 just as in the proof of Theorem 4.4 of [2]. □

Now we analyze the situation on lobe I and show:

**Theorem 4.4.** *On lobe I, we have*

$$\Gamma_4 > \Gamma_2 > 0 > \Gamma_3.$$

*Hence three of the vorticities are positive and one is negative on lobe I.*

*Proof.* The inequality $\Gamma_2 > 0$ follows again from Lemma 4.2. On lobe I, $r_{23} < r_{34}$ and the equation $F_5 = 0$ from (2.19) imply that $\Gamma_3 < 0$. Hence to finish the proof, we only need to show that $\Gamma_4 > \Gamma_2$ on this lobe of the bowtie.

We begin from the equations $F_4 = 0$ and $F_6 = 0$ from (2.19). Solving for $\Gamma_2, \Gamma_4$ and multiplying, we have

$$\Gamma_2 \Gamma_4 = \frac{r_{23} r_{24}^2 r_{34}}{r_{13}^2 r_{14}} \cdot \frac{(r_{13}^2 - r_{14}^2)}{(r_{24}^2 - r_{23}^2)} \cdot \frac{(r_{13}^2 - 1)}{(r_{24}^2 - r_{34}^2)}.$$

We will show first that $\Gamma_2\Gamma_4 > 0$. From (2.3), we also have

$$\frac{(r_{13}^2 - r_{14}^2)}{(r_{24}^2 - r_{23}^2)} = \frac{(r_{14}^2 - r_{34}^2)(r_{13}^2 - 1)}{(1 - r_{23}^2)(r_{24}^2 - r_{34}^2)}. \tag{4.2}$$

Substituting into the previous equation we have

$$\Gamma_2\Gamma_4 = \frac{r_{23}r_{24}^2 r_{34}}{r_{13}^2 r_{14}} \cdot \frac{(r_{14}^2 - r_{34}^2)}{(1 - r_{23}^2)} \cdot \left(\frac{r_{13}^2 - 1}{r_{24}^2 - r_{34}^2}\right)^2.$$

Hence the sign of $\Gamma_2\Gamma_4$ is determined by the sign of the factor $r_{14}^2 - r_{34}^2$.

By rearranging (2.2) and (2.3) (with $r_{12} = 1$), we obtain the following equations

$$\frac{(r_{14}^2 - r_{34}^2)}{(1 - r_{23}^2)} = \frac{(r_{13}^2 - r_{34}^2)(r_{24}^2 - r_{14}^2)}{(r_{13}^2 - r_{23}^2)(r_{24}^2 - 1)} = \frac{(r_{13}^2 - r_{14}^2)(r_{24}^2 - r_{34}^2)}{(r_{13}^2 - 1)(r_{24}^2 - r_{23}^2)}. \tag{4.3}$$

In the rightmost expression in (4.3), all of the factors except $r_{13}^2 - 1$ are known to be positive by equations (2.11), (2.13), and (2.14). Similarly from (2.12) and $r_{23} < r_{34}$, the factors $r_{13}^2 - r_{34}^2$ and $r_{13}^2 - r_{23}^2$ in the middle product are also positive.

We consider the following possible cases. If $r_{24}^2 - r_{14}^2$ and $r_{24}^2 - 1$ have *the same sign*, then $\Gamma_2\Gamma_4 > 0$ and we are done.

On the other hand we claim that the case where these factors have opposite signs, so $r_{24}^2 - 1 < 0$ but $r_{24}^2 - r_{14}^2 > 0$, is not possible for a four-vortex relative equilibrium (even though these relations are certainly possible for a cyclic quadrilateral). We note that in this remaining potential “bad” case, from (4.3), we have $r_{13}^2 - 1 < 0$, so the edge lengths are ordered as follows:

$$r_{12} = 1 > r_{13} > r_{24} > r_{34} > r_{14} > r_{23}. \tag{4.4}$$

We will show that this is incompatible with the equation $F_3 = 0$, but in the rearranged form given in (2.4).

Denote the factors in that equation as $ABC - abc = 0$. Under the assumptions that the lengths are ordered as in (4.4), we see

$$A = r_{24}^2 - r_{23}^2 > a = r_{34}^2 - r_{23}^2 > 0.$$

We claim that it is also true that $BC > bc > 0$, so the equation $ABC - abc = 0$ cannot hold. First, $BC > 0$ and $bc > 0$ by (4.4). Expand out the products in $BC - bc$, noting one cancellation, to obtain

$$r_{13}^2 r_{24}^2 + r_{14}^2 + r_{34}^2 r_{14}^2 - r_{13}^2 r_{14}^2 - r_{34}^2 - r_{14}^2 r_{24}^2.$$

By Ptolemy’s theorem from (2.8) we can substitute for the first term and simplify to obtain

$$BC - bc = r_{14}^2(r_{23}^2 + r_{34}^2 + 1 - r_{13}^2 - r_{24}^2) + 2r_{14}r_{23}r_{34}.$$

By the Law of Cosines as before we have

$$r_{23}^2 + r_{34}^2 = r_{24}^2 + 2r_{23}r_{34}\cos(\theta_3),$$

where $\theta_3$ is the interior angle of the quadrilateral at vertex 3. Hence

$$BC - bc = r_{14}^2(1 - r_{13}^2) + 2r_{14}r_{23}r_{34}(1 + r_{14}\cos(\theta_3)) > 0.$$

This shows that this case cannot occur. Hence the product $\Gamma_2\Gamma_4 > 0$ and in addition $r_{14}^2 - r_{34}^2 > 0$.

It remains to show that $\Gamma_4 > \Gamma_2$. By (2.15) and (2.17),

$$\frac{\Gamma_4}{\Gamma_2} = \frac{r_{34}r_{14}}{r_{23}} \cdot \frac{(r_{13}^2 - 1)(r_{24}^2 - r_{23}^2)}{(r_{13}^2 - r_{14}^2)(r_{24}^2 - r_{34}^2)}.$$

As noted above, from (2.3) (with $r_{12} = 1$), we obtain

$$\frac{(r_{13}^2 - 1)(r_{24}^2 - r_{23}^2)}{(r_{13}^2 - r_{14}^2)(r_{24}^2 - r_{34}^2)} = \frac{(1 - r_{23}^2)}{(r_{14}^2 - r_{34}^2)}. \tag{4.5}$$

Hence

$$\frac{\Gamma_4}{\Gamma_2} = \frac{r_{34} r_{14}(1 - r_{23}^2)}{r_{23}(r_{14}^2 - r_{34}^2)}.$$

Note that both the numerator and the denominator are positive by the argument showing $\Gamma_2\Gamma_4 > 0$. We subtract the denominator in the last expression from the numerator and factor to obtain

$$(r_{34} - r_{14}r_{23})(r_{14} + r_{23}r_{34})$$

The first factor is positive since $r_{34} > r_{23}$ on lobe I and $r_{14} < 1$. The second factor is automatically positive since the $r_{ij}$ are distances. Hence $\Gamma_4 > \Gamma_2$ and the proof is complete. □

## 5. Further constraints on the $q_i$ and the $\Gamma_i$

We have already seen that, as in the Newtonian case, not every cyclic quadrilateral can appear in a relative equilibrium of four vortices; there are additional geometric constraints imposed by (2.1). The following lemma is inspired by the proof of Conley's Perpendicular Bisector Theorem for Newtonian central configurations from [10] and gives another type of constraint. To our knowledge, this sort of argument has not been used before for vortices and this sort of approach could be useful in other situations. However, the fact that the $\Gamma_i$ can be positive or negative makes it somewhat difficult to foresee the circumstances where something of this sort might be used (other than for cases where it is assumed that all the $\Gamma_i$ are positive, for instance). We continue to assume that the positions of the vortices are labeled in sequential order around the circumscribed circle, $r_{12} = 1$ is the longest exterior side of the quadrilateral, $r_{23} \leq r_{14}$, and $\Gamma = 1$.

**Lemma 5.1.** *Let $L$ be the perpendicular bisector of the chord of the circle connecting $q_2$ and $q_3$. Then $q_1$ and $q_4$ lie on opposite sides of $L$. In particular, the arc from $q_1$ to $q_2$ along the circle not containing $q_3$ and $q_4$ is less than a semicircle.*

*Proof.* We begin with the observation that, by Theorems 4.3 and 4.4, $\Gamma_1 = 1$ and $\Gamma_4 > 0$ have the same sign in all of our relative equilibria. From (1.1) with $i = 2, 3$ we have the equations:

$$\begin{aligned}
\Gamma_1 \frac{q_2 - q_1}{r_{12}^2} + \Gamma_3 \frac{q_2 - q_3}{r_{23}^2} + \Gamma_4 \frac{q_2 - q_4}{r_{24}^2} &= -\lambda(q_2 - c) \\
\Gamma_1 \frac{q_3 - q_1}{r_{13}^2} + \Gamma_2 \frac{q_3 - q_2}{r_{23}^2} + \Gamma_4 \frac{q_3 - q_4}{r_{34}^2} &= -\lambda(q_3 - c).
\end{aligned}$$

Subtracting these two equations and rearranging, we see that the vector

$$\Gamma_1 \left( \frac{q_2 - q_1}{r_{12}^2} - \frac{q_3 - q_1}{r_{13}^2} \right) + \Gamma_4 \left( \frac{q_2 - q_4}{r_{24}^2} - \frac{q_3 - q_4}{r_{34}^2} \right) \tag{5.1}$$

is a scalar multiple of $q_2 - q_3$. Let $v$ be a unit vector orthogonal to $q_2 - q_3$. The standard inner (dot) product of $v$ and $q_2 - q_3$ is $\langle v, q_2 - q_3 \rangle = 0$. Hence

$\langle v, q_2 - q_1 \rangle = \langle v, q_3 - q_1 \rangle$ and $\langle v, q_2 - q_4 \rangle = \langle v, q_3 - q_4 \rangle$. Call the first of these scalars $d_1$ and the second $d_4$. Then taking the inner product of (5.1) and $v$ we obtain

$$\Gamma_1 d_1 \left( \frac{1}{r_{12}^2} - \frac{1}{r_{13}^2} \right) + \Gamma_4 d_4 \left( \frac{1}{r_{24}^2} - \frac{1}{r_{34}^2} \right) = 0. \tag{5.2}$$

We claim that this relation can only hold when $q_1$ and $q_4$ lie on opposite sides of $L$. Note that $\frac{1}{r_{12}^2} - \frac{1}{r_{13}^2}$ (respectively, $\frac{1}{r_{24}^2} - \frac{1}{r_{34}^2}$) is zero only if $q_1$ (respectively, $q_4$) lies on the perpendicular bisector $L$. Moreover the sign is positive if $q_1$ (respectively, $q_4$) lies in the half plane bounded by $L$ and containing $q_2$ and negative on the half plane containing $q_3$. On the other hand, $d_1$ and $d_4$ both have the same sign since $q_1$ and $q_4$ lie in the same half-plane bounded by the chord through $q_2$ and $q_3$. Hence the only way the left side of (5.2) can cancel to zero is if $q_1$ and $q_4$ lie on opposite sides of $L$. □

**Theorem 5.2.** *In all of our relative equilibria,* $\Gamma_2 \leq 1$.

*Proof.* In a cyclic quadrilateral, it is a standard fact that the angle between an exterior side and a diagonal is equal to the angle between the opposite side and the other diagonal. It follows that the four triangles formed by the two diagonals and the exterior sides are similar in pairs. In particular the angle at $q_4$ in the triangle formed by $q_1, q_2, q_4$ and the angle at $q_3$ in the triangle formed by $q_1, q_2, q_3$ are equal. Denote this angle by $\theta$. By Lemma 5.1, $\theta < \pi/2$, so $\cos(\theta) > 0$. By the Law of Cosines in these triangles,

$$\begin{aligned} r_{13}^2 + r_{23}^2 &= r_{12}^2 + 2r_{13}r_{23}\cos(\theta) \\ r_{14}^2 + r_{24}^2 &= r_{12}^2 + 2r_{14}r_{24}\cos(\theta). \end{aligned}$$

By Lemma 4.1, $r_{13}r_{23} \leq r_{14}r_{24}$, and hence since $\cos(\theta) > 0$ it follows that $r_{13}^2 + r_{23}^2 \leq r_{14}^2 + r_{24}^2$. Hence $r_{13}^2 - r_{14}^2 \leq r_{24}^2 - r_{23}^2$ and the statement to be proved follows since each of the three factors in the product giving $\Gamma_2$ in (4.1) is $\leq 1$. □

It follows from this result that $\Gamma_1 = 1 \geq \Gamma_2 \geq \Gamma_4 \geq \Gamma_3 > 0$ on lobe II from the previous section. On lobe I we have $\Gamma_4 > \Gamma_2 > 0 > \Gamma_3$, but at present we do not see how to get good bounds on $\Gamma_4$ or $\Gamma_3$.

## 6. The kite configurations

We call a convex quadrilateral a kite if two opposite vertices lie on an axis of symmetry of the configuration (see Figure 6). Thus a co-circular relative equilibrium forms a kite if and only if one pair of opposite vortices lie on the diameter of the circumscribed circle. There are also kites that are not co-circular, but we will not consider them. In the following, we will assume as in Figure 6 that the axis of symmetry passes through vortices 1 and 3.

The definition of a kite implies that adjacent sides are equal for the two vortices that lie on the diameter of the circle. Thus the conditions $r_{12} = r_{14} = 1$ and $r_{23} = r_{34}$ hold. For any kite inscribed in a circle, each side of the line of symmetry forms a right triangle. This gives us the Pythagorean relation

$$r_{13}^2 = 1 + r_{34}^2. \tag{6.1}$$

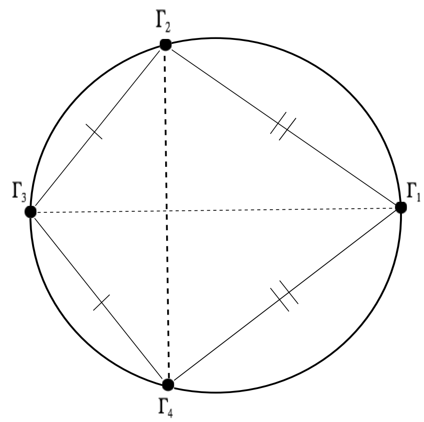


Figure 6: Kite configuration with line of symmetry through vortices 1 and 3

To analyze this case, we will use (2.19), but with $F_5 = 0$ replaced by the equivalent form $F_5' = 0$ from (2.18). We will make use of Gröbner bases for the ideals generated by these polynomials. See [3] for general background on this algebraic technique. Equations for the kite configurations are obtained by substituting $r_{14} = 1$ and $r_{23} = r_{34}$. We adjoin an additional equation

$$1 - tr_{13}r_{24}r_{34}\Gamma_2\Gamma_3\Gamma_4$$

to force the variables appearing there to be nonzero. Using `sage` ([12]), we compute a Gröbner basis for the substituted ideal with respect to the lexicographic order with the variables ordered as follows:

$$t > r_{13} > r_{24} > r_{34} > \Gamma_2 > \Gamma_3 > \Gamma_4.$$

The resulting Gröbner basis contains 24 polynomials, one of which depends only on $\Gamma_3, \Gamma_4$. After factoring, we see that this polynomial is

$$(4\Gamma_4^2 + \Gamma_4\Gamma_3 + \Gamma_4 - 2\Gamma_3)(-4\Gamma_4^2 + \Gamma_4\Gamma_3 + \Gamma_4 + 2\Gamma_3). \tag{6.2}$$

The next polynomial in the Gröbner basis is

$$\Gamma_2 - \Gamma_4,$$

which shows that $\Gamma_2 = \Gamma_4$ for all kite configurations, as we expect from the symmetry.

The real vanishing locus of each of the two factors in (6.2) is a hyperbola in the $\Gamma_3, \Gamma_4$ plane and each of these equations can be solved for $\Gamma_3$ in terms of $\Gamma_4$:

$$\Gamma_3 = \frac{\mp 4\Gamma_4^2 - \Gamma_4}{\Gamma_4 \mp 2} \tag{6.3}$$

(the $-$ sign gives the solution of the equation from the left-hand factor in (6.2) and the $+$ gives the solution of the equation from the right-hand factor).

Adjoining each factor in (6.2) to the ideal individually and computing Gröbner bases again, all of the other variables can be expressed in terms of $\Gamma_4$. From the system using the left-hand factor in (6.2), for instance, we obtain

$$\begin{aligned} r_{34}^2 &= \frac{3\Gamma_4}{\Gamma_4 - 2}, \\ r_{24}^2 &= \frac{6\Gamma_4}{2\Gamma_4 - 1}, \\ r_{13}^2 &= \frac{4\Gamma_4 - 2}{\Gamma_4 - 2}. \end{aligned}$$

All of the right sides must be positive since $r_{ij}$ must be nonzero and real. In addition, $r_{34} \leq 1$ forces $-1 \leq \Gamma_4 \leq 0$. However since $\Gamma_4 > 0$ on the interior of lobes I and II of the bowtie surface from Theorems 4.4 and 4.3, we see that the left-hand factor from (6.2) is satisfied only for points on the surface $F_3 = 0$ with $r_{23} > r_{14}$.

With the right-hand factor in (6.2), we obtain

$$\begin{aligned} r_{34}^2 &= \frac{3\Gamma_4}{\Gamma_4 + 2}, \\ r_{24}^2 &= \frac{6\Gamma_4}{2\Gamma_4 + 1}, \\ r_{13}^2 &= \frac{4\Gamma_4 + 2}{\Gamma_4 + 2}. \end{aligned} \tag{6.4}$$

(The last equation also follows from (6.1).) Now the equation for $r_{34}^2$ shows that to get $0 < r_{34} \leq 1$, we must have $0 < \Gamma_4 \leq 1$. Using the + signs in (6.3), it follows that $\Gamma_3 < 0$ for $0 < \Gamma_4 < \frac{1}{4}$ and $\Gamma_3 > 0$ for $\frac{1}{4} < \Gamma_4 \leq 1$. The points with $\Gamma_3 < 0$ form one of the boundary curves of lobe I of the bowtie surface considered above, and the points with $\Gamma_3 > 0$ give one boundary curve of lobe II. When $\Gamma_4 = \frac{1}{4}$, it follows that $r_{34} = \frac{1}{\sqrt{3}}$, and the corresponding configuration is the symmetric degenerate configuration mentioned before: an equilateral triangle configuration with $\Gamma_1 = 1$, $\Gamma_2 = \Gamma_4 = \frac{1}{4}$, and an additional vortex with $\Gamma_3 = 0$. When $\Gamma_4 = 1$, we have a geometric square configuration with all exterior sides equal to 1, diagonals equal to $\sqrt{2}$, and all vorticities $\Gamma_i = 1$.

We have proved the following statements.

**Theorem 6.1.** *There is exactly one kite configuration corresponding to each point on the intersection of the bowtie surface $F_3 = 0$ and the plane given by $r_{23} = r_{34}$. These configurations are parametrized by the value of the vorticity $\Gamma_4$ with $0 < \Gamma_4 \leq 1$ as in* (6.4)*. The other vorticities are $\Gamma_2 = \Gamma_4$ and $\Gamma_3 = \frac{4\Gamma_4^2 - \Gamma_4}{\Gamma_4 + 2}$. The values $0 < \Gamma_4 \leq \frac{1}{4}$ give the portion of the boundary curve in the closure of lobe I and the values $\frac{1}{4} \leq \Gamma_4 \leq 1$ give the portion of the boundary curve in the closure of lobe II.*

## 7. The isosceles trapezoid configurations

We will call a convex quadrilateral possessing a line of symmetry passing through the midpoints of two opposite edges an isosceles trapezoid. Any such quadrilateral has a circumscribed circle. If we label the vertices as in Figure 7, then the equal pairs of distances are $r_{13} = r_{24}$ and $r_{14} = r_{23}$. The corresponding 4-vortex relative equilibria have been described already in §7 of [8]. Hence we will only briefly discuss how the results of Hampton, Roberts and Santoprete can be recovered with our setup. To analyze this case, we will use (2.19). Equations for the isosceles

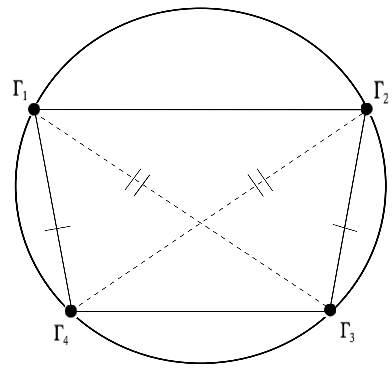


Figure 7: An isosceles trapezoid

trapezoid configurations are obtained by substituting $r_{23} = r_{14}$ and $r_{24} = r_{13}$. We adjoin an additional equation

$$1 - t r_{14} r_{13} r_{34} \Gamma_2 \Gamma_3 \Gamma_4$$

to force the variables appearing there to be nonzero. Using `sage` ([12]), we compute a Gröbner basis for the substituted ideal with respect to the lexicographic order with the variables ordered as follows:

$$t > r_{14} > r_{13} > r_{34} > \Gamma_3 > \Gamma_4 > \Gamma_2.$$

The resulting Gröbner basis contains 35 polynomials. In factored form, the equations from the polynomials with the three smallest lex leading terms are

$$\begin{aligned}(\Gamma_2 - 1)(r_{34} + 1) &= 0,\\ (\Gamma_4 - 1)(\Gamma_3 - \Gamma_4)(r_{34} + 1) &= 0,\\ (r_{34} - 1)(r_{34} + 1)(\Gamma_3 - \Gamma_4) &= 0.\end{aligned}$$

The first implies that $\Gamma_2 = 1$, since $r_{34} > 0$. Similarly, the second implies either $\Gamma_4 = 1$ or $\Gamma_4 = \Gamma_3$ and the third implies $r_{34} = 1$ or $\Gamma_4 = \Gamma_3$. If $r_{34} = 1$, then the configuration must be a geometric square and all the $\Gamma_i = 1$. Hence we see the symmetry of the vorticities directly from the form of the Gröbner basis polynomials.

From the subsequent polynomials in the basis, we can solve for the remaining distances in terms of $\Gamma_3$ with this triangular form system:

$$\begin{aligned} r_{34}^2 &= \frac{2\Gamma_3 + \Gamma_3^2}{2\Gamma_3 + 1},\\ r_{13}^2 &= \frac{\Gamma_3 r_{34}^2 - r_{34}}{\Gamma_3 - r_{34}},\\ r_{14}^2 &= \frac{\Gamma_3 r_{34}^2 + 2r_{13}^2 - \Gamma_3 - 2}{2r_{13}^2 - 2}.\end{aligned} \tag{7.1}$$

From the first equation here, we see that $0 < r_{34} \leq 1$ only when $-2 < \Gamma_3 \leq -1$ or $0 < \Gamma_3 \leq 1$. The last equation then shows $r_{14}^2 > 0$ only when $0 < \Gamma_3 \leq 1$.

**Theorem 7.1.** *There is exactly one isosceles trapezoid configuration corresponding to each point on the intersection of the bowtie surface $F_3 = 0$ and the plane given by $r_{14} = r_{23}$. With the labeling in Figure 7, these configurations are parametrized by the value of the vorticity $\Gamma_3$ with $0 < \Gamma_3 \leq 1$ as in* (7.1)*. The point with $\Gamma_3 = 1$ corresponds to the geometric square configuration.*

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