Mathematics 392–Seminar in Computational Commutative Algebra
Course Information–Spring 2019

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- *Course Homepage:* `http://mathcs.holycross.edu/~little/Sem2019/SemHome.html`
- *Office Hours:* M 2-4pm, T 8-10am, W 11am-12noon, R 1:30-3:30pm, F 11am-12noon, and by appointment.

Course Description and Topics to be Covered

The Seminar in Computational Commutative Algebra is a one-semester advanced topics course, leading up to a significant project assignment that will be your major focus during the last third of the semester. For this reason, it satisfies the Project Course requirement for the Mathematics major.

Commutative algebra is the study of a class of algebraic structures called *commutative rings* and other structures that are built using them. A ring is a set R with two operations $+, \cdot$ that satisfy all of the usual rules of algebra, except that the multiplication operation need not be commutative, and some nonzero elements may not have multiplicative inverses. (In fact, it's even possible to consider rings having no multiplicative identity element, but we'll always assume there is one.)

The fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all examples of rings, but they have extra properties not shared with other examples. A more typical example of a ring is the ring of integers, $\mathbb{Z}$, that you studied in depth in Mathematical Structures (MATH 243). The set of all $n \times n$ matrices with real entries, $M_{n\times n}(\mathbb{R})$, that you studied in Linear Algebra (MATH 244) is also a ring.

A *commutative ring* is a ring in which the multiplication operation is commutative. So for instance $M_{n\times n}(\mathbb{R})$ is not a commutative ring: We can add and multiply square matrices of the same size, but matrix multiplication is not commutative—recall that there are matrices A, B for which $AB \neq BA$. On the other hand, $\mathbb{Z}$ is a commutative ring. Other especially important examples of commutative rings are polynomial rings in one and several variables (and also some closely related rings of rational functions and formal power series).

One reason polynomial rings and their relatives are particularly interesting and worthy of intensive study is that there are many links between the algebra of these rings and the *geometry* of *algebraic varieties*–sets defined as the solutions of systems of polynomial equations. Moreover, the algebra and geometry of solving polynomial equations is important in a wide range of applications.

In this course we will pursue this relation is some depth, together with some relatively recent algorithmic/computational techniques (Gröbner bases) implemented in computer algebra systems such as Maple, Sage, and Magma. To give the idea in a nutshell, an *ideal* I in a ring R is a subset that is closed under sums and also closed under multiplication by all elements of R. A Gröbner basis for an ideal is a special generating set for an ideal that generalizes the row-reduced echelon form for a system of linear polynomials, and also the greatest common divisor of a collection of polynomials in one variable. Each ideal I has many different Gröbner bases, since the definition involves the choice of a *monomial order* – a way of ordering the terms in polynomials that is consistent with the multiplication of polynomials.

If we know a Gröbner basis for a given ideal, then we have a tremendous amount of information about it. Gröbner bases with respect to some monomial orders include polynomials where some variables have been eliminated, for instance. Any Gröbner basis lets us test arbitrary polynomials for membership in the ideal via a general polynomial division process, and so forth.

One of our main goals in the first section of the course will be to develop *Buchberger's algorithm* – a procedure that computes a Gröbner basis for an ideal starting from any generating set for the ideal. We will also look at a few of the ways this algorithm, and others based on it, have been used in applications to geometry, robotics, and other areas.

The topics we will be studying are

- Unit I: The basic language of varieties, ideals, and algorithms (about 6 class days)
- Unit II: Gröbner bases (about 8 days)
- Unit III: Elimination theory, resultants, and geometric applications (about 6 class days)
- Unit IV: "The algebra-geometry dictionary" (about 6 class days)
- Unit V: Applications to quotient rings (about 3 class days)
- Unit VI: Applications to robotics and geometric theorem-proving (about 5 days)

There will be an in-class (or evening) midterm exam and the remaining class days will be devoted to presentations on the final projects.

Text

The text for the course is *Ideals, Varieties, and Algorithms*, 4th edition by Cox, Little, and O'Shea. (Catchy title!) *Important Note:* Because of a "deal" negotiated with the publisher, an electronic version of the text is available *at no charge* via a link from the course homepage. You also have the option of purchasing a physical copy if you prefer. We will cover much of the material in Chapters 1-6 this semester and the rest of the book will be one important source of information for the projects. Please feel free at any time to direct any comments, complaints, etc. about the book to me.

Assignments and Grading

The only way to really learn advanced mathematics is to work out and present solutions to challenging problems. Thus the major focus of this course will be a series of problem sets, given out roughly weekly. Some of these may begin with a computer lab session in the Swords 219 Linux lab, and continue to an individual problem set. Over the course of the semester I will ask each student to give two (short–at most 10 minute) oral presentations to the seminar, either on problem solutions or perhaps on the proof of a particular result from the text. The problems will be similar to ones on the problem sets, and I will assign the presentations when the problem set goes out. You will always have adequate lead time to prepare for these presentations and consult with me if necessary.

There will be a midterm exam. Beside the problem sets and the midterm, the other assignment for the course will be a *final project*. You will work on these projects in individually or in pairs, give a presentation to the seminar on the project, and prepare a roughly 15-page project paper based on your topic. The topics for these presentations can be extensions or conclusions of subjects we study in class, or they can deal with applications of the subjects we consider in other areas. I will distribute more information about these projects later in the semester.

Your final grade for the seminar will be computed using the following weight factors:

- Weekly problem sets – 30% of final grade
- Problem presentations – 10%
- Midterm Exam – 30% – Tentative date: Friday, March 22 (I'm open to doing this as an evening exam instead; we can discuss that in class)
- Final Project presentation – 10%
- Final Project paper – 20%

I will be keeping your course average in numerical form throughout the semester, and only converting to a letter for the final course grade. The course grade will be assigned according to the following conversion table (also see Note below):

- A – 94 and above
- A- – 90 - 93
- B+ – 87 - 89
- B – 84 - 86
- B- – 80 - 83
- C+ – 77 - 79
- C – 74 - 76
- C- – 70 - 73
- D+ – 67 – 69
- D – 60 - 66
- F – 59 and below.

Note: Depending on how the class as a whole is doing, some downward adjustments of the above letter grade boundaries may be made. No upward adjustments will be made, however. (This means, for instance, that an 85 course average would never convert to a letter grade of B- or below, although it might be a B+ in some circumstances.) If you ever have a question about the grading policy or your standing in the course, don't hesitate to ask me.

Advice On How To Succeed In This Class

A good "work ethic" is key. As you should be able to tell from the course description above, this is going to be a challenging course introducing a lot of new mathematics and applying it in ways that will require you to think about things in new and different ways. You will really need to keep up with the assignments and reading from the text.

Come to class. Unless you are deathly ill, have a genuine family emergency, are away at a game or meet of a college athletic team, etc. plan on showing up here at 10:00 am every Monday, Wednesday, and Friday this semester. Some of the class meetings will be structured around labs and/or student presentations. Your participation is expected and needed for the success of the course!

Take notes and use them. This may seem obvious, but it is worth saying! Used intelligently, your notes can be a valuable resource as you work on the problem sets and prepare for the exam.

Use the texts and class notes actively. Reading about mathematics is not like reading a novel. You will probably need to read and think over things more than once. You may want to work through examples to understand some of the topics that we do.

Set up a regular study schedule and work at a steady pace. It's not easy to play catch-up in a mathematics course. You should expect to budget at least 6 hours in a typical week for work outside of class. The best way to use your time is to do a few problems, some reading from the books, and reviewing of class notes every day.

Most importantly, if you are having difficulty learning something, get help as soon as possible. You can do this by asking questions during class (any time something isn't clear), or seeing me during office hours.

Statement on Academic Integrity

All education is a cooperative enterprise between teachers and students. This cooperation works well only when there is trust and mutual respect between everyone involved. To be become an engaged and advanced learner, you must be able to think and work both independently and in concert with your peers. The College academic honesty policy states: "As an institution devoted to teaching, learning, and intellectual inquiry, Holy Cross expects all members of the College community to abide by the highest standards of academic integrity. Any violation of academic honesty undermines the student-teacher relationship, thereby wounding the whole community. The principal violations of academic honesty are plagiarism, cheating, and collusion.

Plagiarism is the act of taking the words, ideas, data, illustrative material, or statements of someone else, without full and proper acknowledgment, and presenting them as one's own.

Cheating is the use of improper means or subterfuge to gain credit or advantage. Forms of cheating include the use, attempted use, or improper possession of unauthorized aids in any examination or other academic exercise submitted for evaluation; the fabrication or falsification of data; misrepresentation of academic or extracurricular credentials; and deceitful performance on placement examinations. It is also cheating to submit the same work for credit in more than one course, except as authorized in advance by the course instructors.

Collusion is assisting or attempting to assist another student in an act of academic dishonesty."

The full statement on Academic Integrity in the College Catalog is available at

`https://www.holycross.edu/sites/default/files/files/registrar/academic_integrity_policy_0.pdf`

The temptation to engage in an act of academic dishonesty will almost certainly arise, but the chance to possibly enhance a single grade is not worth the loss of your personal integrity. NOTE: If in doubt about what you plan to do or write violates academic honesty policy, please ask Prof. Little for guidance before acting.

Specific Guidelines for this Course

In this course, the midterm examination will be closed-book. No sharing of information with other students in any form will be permitted during that. On the individual problem sets, discussion of the questions with other students in the class and with me during office hours is allowed, *even encouraged*. However, your final problem solutions should be prepared individually and the wording and organization of your final problem solutions should be entirely your own work. Moreover, if you do take advantage of any of the above options for discussion of problems with others, you will be required to state that fact in a footnote accompanying the problem solution. Failure to follow this rule will be treated as a violation of the College's Academic Integrity policy. For the project paper, you will be consulting sources other than the course text, and you will need to include a full reference in a bibliography section at the end of your paper. Information about the acceptable formats for doing this will be distributed with the project assignment.

Policy on Excused Absences

This is available at:

`http://www.holycross.edu/sites/default/files/files/registrar/excused_absence_policy.pdf`

Disability Statement

Any student who feels the need for accommodation based on the impact of a disability should contact the Office of Disability Services to discuss support services available. Once the office receives documentation supporting the request for accommodation, the student would meet privately with Disability Services to discuss reasonable and appropriate accommodations. The office can be reached by calling 508-793-3693 or by visiting Hogan Campus Center, room 215A.

Statement on Diversity and Inclusion

It is my intent that students from all diverse backgrounds and perspectives be well-served by this course, that students' learning needs be addressed both in and out of class, and that the diversity that the students bring to this class be viewed as a resource, strength and benefit. It is my intent to present materials and activities that are respectful of diversity: gender identity, sexuality, disability, age, socioeconomic status, ethnicity, race, nationality, religion, and culture. Your suggestions are encouraged and appreciated. Please let me know ways to improve the effectiveness of the course for you personally, or for other students or student groups.