

Mathematics 392–Seminar in Computational Commutative Algebra  
Lab Day 3/Problem Set 6  
March 11, 2019

*General Information and Ground Rules*

Today’s lab session, and the problems for Problem Set 6 (due in class on Friday, March 15) will focus on two applications of elimination: the *implicitization* problem for parametric surfaces, and finding the *envelope* of a family of curves in the plane. These topics are discussed in sections 3 and 4 of Chapter 3 in “IVA”. For this problem set, *please attach your Maple worksheet file(s) to an email sent to jlittle@holycross.edu to submit your work. Include all answers to the questions below and use text regions for comments, explanations, etc.*

*First Problems/Examples*

I. Consider the parametric surface in  $\mathbb{C}^3$  defined by

$$\begin{aligned}x &= uv + v \\y &= u^2 - 2v \\z &= v^2 + u - v\end{aligned}$$

A. Compute a *lex* Gröbner basis for the ideal

$$I = \langle x - uv - v, y - u^2 + 2v, z - v^2 - u + v \rangle$$

with the variables arranged to eliminate  $u, v$ .

B. Use your answer to part A to find a variety  $V \subset \mathbb{C}^3$  containing the image of the parametrization mapping.

C. Use the Extension Theorem from class on Friday, March 1 to prove that every point of  $V$  is in the image of the parametrization mapping over  $\mathbb{C}$ . (Hints: This is equivalent to showing that every partial solution  $(x, y, z)$  in  $V(I_2) \subset \mathbb{C}^3$  extends to a full solution  $(x, y, z, u, v)$  in  $V(I) \subset \mathbb{C}^5$ . The Gröbner basis is complicated, but you’ll find what you need if you look hard enough!)

II. (Envelopes) A family of curves in the plane is given by an equation  $F(x, y, t) = 0$ . Setting  $t$  equal to a constant gives one of the curves in the family. In this problem, we will study the family defined by the equation:

$$F(x, y, t) = (x - t)^2 + (y - t^2)^2 - 1 = 0$$

From the form, we see that each curve  $F(x, y, t_0)$  is a circle of radius 1 whose center is a point on the parabola  $(t, t^2)$ . We can visualize the curves in such a family in Maple by generating an

*animated* plot where  $t$  is the “animation variable”. It is most convenient to do this in parametric form though, so we will consider the parametric equations:

$$\begin{aligned}x &= t + \cos(\theta) \\ y &= t^2 + \sin(\theta)\end{aligned}$$

where the  $(\cos(\theta), \sin(\theta))$  parametrizes the circle, and the  $(t, t^2)$  give the coordinates of the center.

- Load the `plots` package:

```
with(plots):
```

- Then enter the following command:

```
animate([t+cos(theta), t^2+sin(theta), theta=0..2*Pi], t=-2..2, frames=20);
```

- To “play the animation”, you first left click the mouse over the graph, and select the options (forward or backward, loop or once, etc.) from the toolbar. Then right click the mouse over the graph and select play from the animate option on the plot menu.
- The *envelope* of the family of curves given by  $F(x, y, t) = 0$  is another curve that is *tangent* to each of the curves in the family at the intersection points of the envelope and the family curve. It is found by eliminating  $t$  between the two equations:

$$\begin{aligned}F(x, y, t) &= 0 \\ \frac{\partial}{\partial t}F(x, y, t) &= 0\end{aligned}$$

(This is discussed in detail in §3.4 of “IVA.”) For now, let’s take this as given and find the envelope for the family of curves above.

- A) Define  $F = (x - t)^2 + (y - t^2)^2 - 1$  as above, and compute an appropriate Gröbner basis to find generator(s) of the elimination ideal

$$I_1 = \langle F(x, y, t), \frac{\partial}{\partial t}F(x, y, t) \rangle \cap \mathbb{C}[x, y].$$

In Maple, the partial derivative can be computed with the `diff` command. Consult the online help for documentation and examples.

- B) Use `implicitplot` to generate a picture of the envelope curve for this family of circles (the variety  $V(I_1)$  with  $I_1$  as in part B).
- C) To understand the geometry here in more detail, it will help to generate a composite plot showing the animated family of curves, together with the implicit plot of the envelope, and also the path followed by the centers of the circles in the family. The `display` command in the `plots` package can be used to combine plots in this way. Look up the online help and see if you can figure out how it should work.

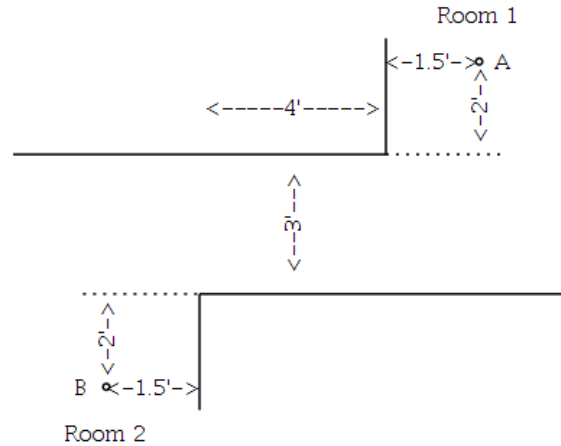


Figure 1: The floor plan of the alien complex (not exactly to scale)

III. You have just discovered the remnants of an ancient alien civilization in a subterranean complex. Near the entrance there are two large rooms connected by a short corridor 3 feet wide and 4 feet long. The floor plan is given in the diagram in Figure 1.

In Room 1, at the point marked A, you have discovered a functional, but very fragile, robot. It has a horizontal cross-section that is a circle of diameter 2 feet. The robot will accept movement commands, but will not let itself be moved by any other means. Your job is to figure out a way (any way) to maneuver the robot from point A to point B, subject to the following constraints:

- 1) The aliens were apparently very clever mathematicians, but the robot's control program is *very limited* – the way you must specify its motion is to give a single polynomial parametric curve  $x = f(t), y = g(t)$  that traces the path of the *center* of the robot's circular cross-section. Only one curve is allowed – no piecewise polynomial paths.
- 2) The robot will be damaged beyond repair if it runs into one of the walls. If it does, you will have destroyed the most amazing archeological discovery of all time. *So be careful!*
- 3) Fortunately, you get to “simulate” possible motions using Maple as often as you like until you find one that works, and then you can give the information to the robot.

For this problem you will need to generate a (section of a) Maple worksheet with graphics showing the rooms and corridor, and a display showing that your path will keep the robot from hitting the walls at any point. And, you will *give a justification for why your path will not run the robot into the walls*. (By this I mean more evidence than the picture – for instance, if feasible, you could find

the point on the envelope that is closest to the walls, and show that the distance at the closest point is strictly positive.)

### *Suggestions*

- 1) A good first step will be to set up a convenient coordinate system. This will let you set up plots showing the rooms, etc. You can also experiment and determine some paths from point A to point B to try.
- 2) The discussion of envelopes from question II above is relevant here!
- 3) *Important Note:* You will need to be somewhat clever in how you apply the techniques we have discussed – if you use a “brute force” method, to compute the envelope for some “obvious” choices of the path, then you can easily end up giving Maple a Gröbner basis calculation that will use up an hour of calculation and tie up your computer without producing any results(!) There are several things you can experiment with:
  - the monomial order you use (look at the online help for MonomialOrder)
  - you can reduce the number of variables by looking at one “slice”  $x = \text{const}$  or  $y = \text{const}$  at a time. This generally speeds things up *a lot*, but you will need to look at several of these computations to make sure the robot follows an acceptable path.
  - However you approach the problem, you will need *give a justification for why your path will not run the robot into the walls.*