MATH 392 – Seminar in Computational Commutative Algebra
Solutions for Problem Set 8
April 5, 2019

4.3/6. Let $I_1, \ldots, I_r$ and J be ideals in $k[x_1, \ldots, x_n]$.

a. $\subseteq$: Let $f \in (I_1 + I_2)J$. Then by definition, f is a *sum of terms* of the form $g \cdot h$ where $g \in I_1 + I_2$ and $h \in J$. Then by definition we have $g = g_1 + g_2$ with $g_i \in I_i$, $i = 1, 2$. By the distributive law, f is a sum of terms of the form $g_1 h + g_2 h$. Regrouping the terms, we can write this sum as a sum of elements of $I_1 J$ plus a sum of elements of $I_2 J$. This shows $f \in I_1 J + I_2 J$.
$\supseteq$: Now assume that $f \in I_1 J + I_2 J$. Then $f = g + h$ with $g \in I_1 J$ and $h \in I_2 J$. But note that $I_1 \subseteq I_1 + I_2$ and $I_2 \subseteq I_1 + I_2$. Hence $g \in (I_1 + I_2)J$ and $h \in (I_1 + I_2)J$. This implies $f \in (I_1 + I_2)J$ as well, since this is an ideal, and hence closed under sums.

b. In office hours with a number of you, I suggested using a proof by induction on $r \geq 1$ rather than plunging into a direct proof. That is certainly possible, although still somewhat messy notationally. *By far the slickest proof*, however, comes by doing an induction on m(!) taking r as fixed but arbitrary. Here is how it works. The base case is $m = 1$, and then there is nothing to prove since $I_1 \cdots I_r = I_1 \cdots I_r$ is obviously true. Now assume that $(I_1 \cdots I_r)^{m-1} = I_1^{m-1} \cdots I_r^{m-1}$, and consider $(I_1 \cdots I_r)^m$. By definition,
$$\begin{aligned}(I_1 \cdots I_r)^m &= (I_1 \cdots I_r)^{m-1}(I_1 \cdots I_r)\\ &= I_1^{m-1} \cdots I_r^{m-1} I_1 \cdots I_r\\ &= I_1^m \cdots I_r^m.\end{aligned}$$
We used the induction hypothesis to get from the first line to the second, and then used the fact that the product in the polynomial ring is commutative and hence that ideal products are also commutative.

Moral here: No one is going to give you a prize in mathematics for working harder than you need to! A big part of mathematics is actually *being lazy in creative ways*(!)

8. See Maple worksheet printout attached.

9. First we want to show that $\sqrt{IJ} = \sqrt{I \cap J}$ for ideals in $k[x_1, \ldots, x_n]$ with k an arbitrary field.

$\subseteq$: This follows immediately from a fact that we noted in class: $IJ \subseteq I \cap J$. The inclusion is then preserved under taking radicals.
$\supseteq$: If $f \in \sqrt{I \cap J}$, then there is some integer $m \geq 1$ such that $f^m \in I \cap J$. This implies $f^m \in I$ and $f^m \in J$. But then $f^{2m} = f^m \cdot f^m \in IJ$ and hence $f \in \sqrt{IJ}$ by definition. It follows that $\sqrt{I \cap J} \subseteq \sqrt{IJ}$.
Then for the last parts of the question, let $I = J = \langle x \rangle$. It is easy to see that I, J are radical ideals, but $IJ = \langle x^2 \rangle$ is not radical (because $x \in \sqrt{IJ}$, but $x \notin IJ$). Similarly, $\sqrt{IJ} = \sqrt{\langle x^2 \rangle} = \langle x \rangle$. But this is different from $\sqrt{I}\sqrt{J} = \langle x \rangle \langle x \rangle = \langle x^2 \rangle$.

11. By the definition, ideals I, J in $k[x_1, \ldots, x_n]$ are comaximal if $I + J = \langle 1 \rangle = k[x_1, \ldots, x_n]$.

a. We need to show that, under the assumption $k = \mathbf{C}$, I, J are comaximal if and only if $V(I) \cap V(J) = \emptyset$.
$\Rightarrow$: If I, J are comaximal, then $1 \in I + J$, so $V(I + J) = \emptyset$. But we also know that $V(I + J) = V(I) \cap V(J)$ for all ideals, so then $V(I) \cap V(J) = \emptyset$. (**Note:** This implication did not use the hypothesis that $k = \mathbf{C}$. It is actually true over any field k.)
$\Leftarrow$: Now assume that $V(I) \cap V(J) = \emptyset$. As in the other implication, this says $V(I + J) = \emptyset$. But now since we are working over $\mathbf{C}$, the Weak Nullstellensatz implies $I + J = \mathbf{C}[x_1, \ldots, x_n]$. Hence I, J are comaximal.
This statement is not true without the assumption that k is algebraically closed. For instance, if $I = \langle x^2 + 1 \rangle = J$ in $\mathbf{R}[x]$, then $V(I) \cap V(J) = \emptyset$ since actually $V(I) = \emptyset = V(J)$ already. But note that $I + J = \langle x^2 + 1 \rangle$ because ideals are closed under sums. This is not the whole ring $\mathbf{R}[x]$ since, for instance $1, x \notin \langle x^2 + 1 \rangle$.

b. Now we assume that I, J are comaximal and we want to show $IJ = I \cap J$.
$\subseteq$: This is always true as we noted above.
$\supseteq$: If I, J are comaximal, then there are $f \in I$ and $g \in J$ such that $f + g = 1$. Take $h \in I \cap J$ and multiply both sides of this equation by h: $hf + hg = h$. Since $finI$ and $h \in J$, the term $hf \in IJ$. Similarly since $h \in I$ and $g \in J$, the term $hg \in IJ$. Since IJ is an ideal, it is closed under sums, and this shows $h \in IJ$.