4.2/15. An *alternate proof*: We need to show that $I = \langle xy, xz, yz \rangle$ is a radical ideal, or equivalently that if $f^m \in I$ for some $m \geq 1$, then $f \in I$.

An equivalent statement is the contrapositive form of this implication:

- If $f \notin I$, then $f^m \notin I$ for *all* $m \geq 1$.

(Note that the existential quantifier "for some" flips to "for all" when we negate the quantified statement.)

Now, to start, the set $G = \{xy, xz, yz\}$ is easily seen to be a Gröbner basis for I by Buchberger's Criterion: The S-polynomials of all pairs equal 0, hence they give zero remainders on division by G. Since the condition of Buchberger's Criterion is satisfied, G is a Gröbner basis for I. (Note: This is true for *any monomial ideal*, in fact. The finite generating set promised by Dickson's Lemma is also a Gröbner basis for the monomial ideal!)

Hence $f \notin I$ means that f contains a term that is not divisible by any of xy, xz, yz. The terms that are not divisible by any of these three monomials are the "pure powers"

$$x^k, y^k, z^k$$

for $k \geq 0$. So if $f \notin I$, say f contains a term cx^k for some k, and *in addition, assume that this is the highest pure power of x appearing in f with $c \neq 0$.* If $m \geq 1$, then when we expand out the power we get

$$f^m = (cx^k + \cdots)^m = c^m x^{km} + \cdots,$$

and the x^{km} must be the highest pure power of x that is contained in f^m. Hence, since $c^m \neq 0$, we have $f^m \neq I$. Since this is true for all $m \geq 1$, we see that $f \notin I$ implies $f^m \notin I$ for all $m \geq 1$, and that is what we wanted to show in this case. The cases where f contains pure powers of y and/or z are similar.