> $with(Groebner)$:

MATH 392 -- Seminar in Computational Commutative Algebra
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From plane Euclidean geometry, recall:

Theorem. The altitudes of a general triangle are concurrent (that is, all three altitudes meet at at one point, called the *orthocenter* of the triangle).

We can ***translate*** the hypotheses and conclusion of the theorem into ***polynomial equations*** by introducing coordinates like this:

- Place the vertices of the triangle at A = (0,0), B = $(u_1, 0)$, $C = (u_2, u_3)$,
- Then construct the intersection points of the altitudes and the opposite sides: $P = (x_1, x_2)$, $Q = (x_3, x_4)$, $R = (x_5, 0)$.

- Let $\mathrm{O} = (x_6, x_7)$ be the intersection of *AP* and *BQ.*
- The hypotheses are that:
  C,B,P are collinear (h_1 below)
  AP is perpendicular to *BC* (h_2 below)
  A,Q,C are collinear (h_3 below)
  BQ is perpendicular to *AC* (h_4 below)
  CR is perpendicular to *AB* (h_5 below)
  O is collinear with *B, Q* (h_6 below)
  O is collinear with *A, P* (h_7 below)

- Then the conclusion says that *CO* is the same as the altitude from *C,* so it is perpendicular to *AB* (g below)

> $h_1 := (x_2 - u_3) \cdot (x_1 - u_1) - x_2 \cdot (x_1 - u_2);$

$$h_1 := (x_2 - u_3)\,(x_1 - u_1) - x_2\,(x_1 - u_2) \qquad \textbf{(1)}$$

> $h_2 := x_2 \cdot u_3 - x_1 \cdot (u_1 - u_2);$

$$h_2 := x_2\,u_3 - x_1\,(u_1 - u_2) \qquad \textbf{(2)}$$

> $h_3 := x_4 \cdot (x_3 - u_2) - x_3 \cdot (x_4 - u_3);$

$$h_3 := x_4\,(x_3 - u_2) - x_3\,(x_4 - u_3) \qquad \textbf{(3)}$$

> $h_4 := x_4 \cdot u_3 - u_2 \cdot (u_1 - x_3);$

$$h_4 := x_4\,u_3 - u_2\,(u_1 - x_3) \qquad \textbf{(4)}$$

> $h_5 := u_2 - x_5;$

$$h_5 := u_2 - x_5 \qquad \textbf{(5)}$$

> $h_6 := (x_3 - u_1) \cdot x_7 - x_4 \cdot (x_6 - u_1);$

$$h_6 := (x_3 - u_1)\,x_7 - x_4\,(x_6 - u_1) \qquad \textbf{(6)}$$

> $h_7 := x_1 \cdot x_7 - x_2 \cdot x_6;$

$$h_7 := x_1\, x_7 - x_2\, x_6 \qquad \textbf{(7)}$$

> $g := x_6 - u_2;$

$$g := x_6 - u_2 \qquad \textbf{(8)}$$

(The conclusion could also be expressed by the same methods as above, but this is equivalent.)

We would say g ***follows directly*** from the hypotheses $h_i,\ i = 1, ..., 7$ if $g \in \sqrt{\langle h_1, \ldots, h_7 \rangle}$
(since then $g \in \boldsymbol{I}(\boldsymbol{V}(h_1, \ldots, h_7))$ (Over C, these would be equal by the Nullstellensatz.)

Does this work here? We use the ***radical membership algorithm*** from p. 185 of (the 4th edition of) "IVA".

> $IdRad := [h_1, h_2, h_3, h_4, h_5, h_6, h_7, 1 - g \cdot y];$

$$IdRad := [(x_2 - u_3)(x_1 - u_1) - x_2 (x_1 - u_2),\ x_2 u_3 - x_1 (u_1 - u_2),\ x_4 (x_3 - u_2) - x_3 (x_4 - u_3),\ x_4 u_3 - u_2 (u_1 - x_3),\ u_2 - x_5,\ (x_3 - u_1) x_7 - x_4 (x_6 - u_1),\ x_1 x_7 - x_2 x_6,\ -(x_6 - u_2) y + 1] \qquad \textbf{(9)}$$

> $BRad := Basis(IdRad, tdeg(x_1, x_2, x_3, x_4, x_5, x_6, x_7, u[1], u[2], u[3], y));$

$$BRad := [x_5 - u_2,\ -y u_2 + y x_6 - 1,\ x_4 u_2 - x_3 u_3,\ -u_1 u_2 + u_2 x_3 + x_4 u_3,\ -u_3 u_1 + x_2 u_1 - x_2 u_2 + u_3 x_1,\ x_1 u_1 - x_1 u_2 - x_2 u_3,\ -x_4 u_1 + u_1 x_7 - x_3 x_7 + x_4 x_6,\ -x_1 x_7 + x_2 x_6,\ -u_3 u_1 - x_2 u_2 + u_3 x_1 - x_1 x_4 + x_2 x_3,\ y u_1 x_4 - y u_1 x_7 - y u_3 x_3 + y x_3 x_7 - x_4,\ -y u_2 x_2 + y x_1 x_7 - x_2,\ u_1^2 u_3,\ u_1 u_3^2 + u_1 u_3 x_4 + u_2 u_3 x_2 - u_3^2 x_1,\ u_1 u_3 x_3 + u_2 u_3 x_1 + u_3^2 x_2,\ u_2 u_3 x_1 + u_3^2 x_2 + u_3 x_3 x_6 + u_3 x_4 x_7,\ -u_1 u_3 x_6 + u_3 x_1 x_6 + u_3 x_2 x_7,\ u_2 u_3 x_1 + u_3^2 x_2 + u_3 x_3^2 + u_3 x_4^2,\ u_3 x_1 x_3 + u_3 x_2 x_4,\ -u_2 u_3 x_1 - u_3^2 x_2 + u_3 x_1^2 + u_3 x_2^2,\ y u_1 u_2 u_3 + y u_2 u_3 x_1 + y u_3^2 x_2 - y u_3^2 x_4 + y u_3 x_4 x_7 + x_3 u_3,\ -y u_1 u_2 u_3 + y u_2 u_3 x_1 + y u_3 x_2 x_7 - u_3 u_1 + u_3 x_1,\ u_1 u_2 u_3^2 + u_2^2 u_3 x_2 - 2 u_2 u_3^2 x_1 - u_3^3 x_2,\ u_1 u_3^3 + u_2^2 u_3 x_1 + 2 u_2 u_3^2 x_2 - u_3^3 x_1,\ u_1 u_2 u_3 x_6 + u_1 u_3^2 x_7,\ u_1 u_3^3 + u_2 u_3^2 x_2 + u_2 u_3 x_2^2 - u_3^3 x_1 - u_3^2 x_1 x_2,\ u_2 u_3 x_1 x_2 + u_3^2 x_2^2,\ y u_1 u_2^2 u_3 + y u_1 u_3^2 x_7 + u_1 u_2 u_3] \qquad \textbf{(10)}$$

Note we do ***NOT*** get [1] as we expected (the theorem is true!). What happened?

We can see by looking at a GB for the ideal generated by the hypotheses:

```
> HB := Basis([h[1], h[2], h[3], h[4], h[5], h[6], h[7]], tdeg(x[1], x[2], x[3], x[4],
     x[5], x[6], x[7], u[1], u[2], u[3])) :
  for i to nops(HB) do
     factor(HB[i]);
  end do;
```

$$x_5 - u_2$$

$$x_4 u_2 - x_3 u_3$$

$$-u_1 u_2 + u_2 x_3 + x_4 u_3$$

$$-u_3 u_1 + x_2 u_1 - x_2 u_2 + u_3 x_1$$

$$x_1 u_1 - x_1 u_2 - x_2 u_3$$

$$-x_4 u_1 + u_1 x_7 - x_3 x_7 + x_4 x_6$$

$$-x_1 x_7 + x_2 x_6$$

$$-u_3 (u_1 x_3 - x_3 x_6 - x_4 x_7)$$

$$-u_3 (u_1 x_6 - x_1 x_6 - x_2 x_7)$$

$$-u_3 (u_1 x_3 - x_3^2 - x_4^2)$$

$$-u_3 (x_1 u_2 + x_2 u_3 - x_1^2 - x_2^2)$$

$$u_1 u_3 x_4 - u_1 u_3 x_7 - u_2 x_2 x_7 - u_3 x_1 x_4 + u_3 x_1 x_7 + u_3 x_2 x_3 - x_1 x_4 x_7 + x_2 x_3 x_7$$

$$-u_3 u_1 (u_1 u_2 - u_2 x_6 - u_3 x_7)$$

$$-u_1^2 u_3 (u_2 - x_6)$$

$$-u_3 (u_1^2 x_3 + u_1 u_3 x_4 - u_1 u_3 x_7 - u_1 x_4 x_7 - u_2^2 x_1 - u_2 u_3 x_2 - u_2 x_2 x_7 + u_3 x_1 x_7)$$

$$u_3 (u_1^2 x_4 - u_1^2 x_7 - u_1 u_2 u_3 - u_1 u_3 x_3 + u_1 x_3 x_7 - u_2^2 x_2 + u_2 u_3 x_1 + u_2 x_1 x_7 + u_3 x_2 x_7)$$

$$-u_3 (u_1 u_2 u_3 + u_1 u_3 x_3 + u_2^2 x_2 - u_2 u_3 x_1 - u_3 x_1 x_3 - u_3 x_2 x_4 - x_1 x_3 x_7 - x_2 x_4 x_7)$$

$$-u_3 (u_1^2 u_2^2 - u_1 u_2 u_3^2 - u_1 u_2 u_3 x_7 - u_2^3 x_1 - 2 u_2^2 u_3 x_2 - u_2^2 x_2 x_7 + u_2 u_3^2 x_1 + 2 u_2 u_3 x_1 x_7 + u_3^2 x_2 x_7)$$

$$-u_3 (u_1^2 u_2 u_3 + u_1 u_2^2 u_3 - u_1 u_3^2 x_7 + u_2^3 x_2 - 2 u_2^2 u_3 x_1 - u_2^2 x_1 x_7 - u_2 u_3^2 x_2 - 2 u_2 u_3 x_2 x_7 + u_3^2 x_1 x_7)$$

$$u_1^2 u_3 (u_1 u_2 - u_2^2 - u_3 x_7)$$

$$-u_3 (u_1^2 u_2 u_3 + u_1 u_2^2 u_3 - u_1 u_3^2 x_7 + u_2^3 x_2 - 2 u_2^2 u_3 x_1 - u_2^2 x_1 x_2 - u_2 u_3^2 x_2 - u_2 u_3 x_2^2 - u_2 u_3 x_2 x_7 - u_2 x_2^2 x_7 + u_3^2 x_1 x_7 + u_3 x_1 x_2 x_7)$$

$$-u_3 (u_1 u_2 u_3^2 + u_2^2 u_3 x_2 + u_2^2 x_2^2 - u_2 u_3^2 x_1 - u_2 u_3 x_1 x_2 - u_2 x_1 x_2 x_7 - u_3 x_2^2 x_7) \quad \textbf{(11)}$$

```
>
```

The 11th element of the GB is $-u_1^2 u_3 (u_2 - x_6)$, which is a multiple of the conclusion,

but includes as a factor a monomial in the *u-variables.* Recall those are the coordinates of the arbitrary
points *A, B, C -- the vertices of the triangle.* What's going on here is that there are certain"degenerate"
configurations of those points for which we don't have an "honest triangle," but which still give solutions
of the polynomial form of the geometric hypotheses:

- The equation $u\_1 = 0$ means $B = A$;
- The equation $u\_3 = 0$ means *A, B,* and *C* all lie along the line *y* = 0

The idea is that we want to ***remove*** those degenerate configurations from consideration! A first thing to do to accomplish that is to treat the *u*-variables as *invertible*. In algebraic terms, we do the GB computation over a different coefficient field, namely < (*u_1,u_2,u_3*) or C(*u_1, u_2, u_3*) (rational functions) rather than over the field of constants. Here is the computation from the radical membership algorithm done this alternate way:

```
> Basis(IdRad, tdeg(x_1, x_2, x_3, x_4, x_5, x_6, x_7, y));
```

$$[1] \qquad (12)$$

We say in a case like this that ***the conclusion follows* generically *from the hypotheses,***
since it follows whenever $u\_1 \neq 0$ and $u\_3 \neq 0$ (and these conditions are necessary to say we have an actual triangle!)

```
> HB := Basis([h[1], h[2], h[3], h[4], h[5], h[6], h[7]], tdeg(x[1], x[2], x[3], x[4],
      x[5], x[6], x[7])) :
```

```
> NormalForm(g, HB, tdeg(x[1], x[2], x[3], x[4], x[5], x[6], x[7]));
```

$$0 \qquad (13)$$

```
>
```