MATH 392 – Seminar in Computational Commutative Algebra
Information for Midterm Exam

General Information

Still to be determined is whether the exam will be given in class on Friday, March 22 or the previous Thursday evening. Either option is fine with me; we will discuss this in class on Wednesday, March 13.

- If we do the exam Friday, you will have the full period. If we do it Thursday evening, you will have two hours. I reserve the right to make the exam slightly longer (nowhere near twice as long, though) if we decide to do it in the evening.
- This is an individual exam. No sharing of information with others in any form will be permitted while you are taking it.
- You may use a calculator during the exam, but *no other electronic devices.*
- There will be four or five problems (each possibly with a few separate parts). These questions will draw on *the material we have studied* from Chapters 1, 2, and 3 of our text. Some sample exam questions are given later in this document.
- I won't ask you to do lots of computations by hand because we have done almost all of that heavy lifting using Maple, and you won't have that available during the exam. *One exception to that general rule is that you should be prepared to do "reasonable" polynomial divisions in one or several variables, and/or carry out the Euclidean Algorithm to find the gcd of two polynomials in one variable.*
- The exam will focus more on the conceptual background, definitions, key results we have studied, and perhaps understanding what output from Maple for some of the computations we have done means. I may ask you for statements and/or proofs of main results. If a question says "state Theorem X" then you need only give the statement of the Theorem; there is no need to give the proof. Any such question will be the same as one of the parts of the practice problems below.

Some sample practice/review problems

I. Draw or describe in words each of the following varieties.

A) $\mathbf{V}(x^2 + y^2 - 4, x - y + 1)$ in $\mathbf{R}^2$
B) $\mathbf{V}(z - xy)$ in $\mathbf{R}^3$
C) $\mathbf{V}(x^2 + y^2 - 9, z - x^2)$ in $\mathbf{R}^3$.

II. All parts here refer to the polynomial ring $k[x_1, \ldots, x_n]$.

A) Define: I is an ideal.
B) Define: The radical $\sqrt{I}$ of an ideal I.
C) Show that if I is an ideal, then $\sqrt{I}$ is also an ideal.

III.

A) Prove that every ideal I in the polynomial ring $k[x]$ (one variable) is principal (that is, $I = \langle g(x) \rangle$ for some single polynomial).
B) Find $g(x)$ as in part A for the ideal $I = \langle x^3 + 4x - 16, x^2 - 5x + 6\rangle$

IV.

A) Define: $>$ is a monomial order on $k[x_1, \ldots, x_n]$.
B) What property of a monomial order implies that the division algorithm in $k[x_1, \ldots, x_n]$ will always terminate in a finite number of steps? (That is, the algorithm terminates no matter what the divisors $f_1, \ldots, f_s$ and the dividend f are.)
C) What are the quotients and the remainder on division of $f = x^3y - 4y$ by $f_1 = x^2 + xy$ and $f_2 = y - 1$ using the lex order with $x > y$? Using the lex order with $y > x$?

V.

A) State Dickson's Lemma.
B) Define: G is a Gröbner basis for an ideal I with respect to a monomial order $>$.
C) Prove that Gröbner bases exist for every ideal I in $k[x_1, \ldots, x_n]$ and with respect to every monomial order. (Hint: Use Dickson's Lemma; state the definition of any auxiliary ideals you need in the proof.)
D) Deduce from C that every ideal in $k[x_1, \ldots, x_n]$ has a finite basis (the Hilbert Basis Theorem).
E) Is $\{x^2y - xy + 1, xy^2 + 1\}$ a Gröbner basis for the ideal generated by these two polynomials? Why or why not? State any key results you are using to derive your conclusion.

VI.

A) Show that if G is a Gröbner basis for an ideal I with respect to the monomial order $>$ and $f \in I$, then the remainder on division of f by G using $>$ is *zero*. Do your proof as a formal proof by induction on an integer m counting the steps in the division.
B) Define: The ℓth elimination ideal I_ℓ of an ideal $I \subseteq k[x_1, \ldots, x_n]$.
C) State and prove the *Elimination Theorem*.
D) State the *Extension Theorem*.

VII. For all parts of this question, refer to the Maple output from the course homepage.
A) From the computation for this part, what can you conclude about $f = x^3 + y^3 + z^3 - 2z^2 + 2y - 33z$?
B) From the computation for this part, determine Gröbner bases for the elimination ideals I_1 and I_2 for $I = \langle y^3 + z^3 + x^2 - 1, -y^2 + x + z + 1, x^2 - y^2 - z + 1\rangle$.
C) Continuing from part B, what does the Extension Theorem tell you about points in the variety $\mathbf{V}(I)$ in $\mathbf{C}^3$? Explain step by step and say how many points there are on the variety for each z solving the equation $BB[1] = 0$. Note: A consequence of the Fundamental Theorem of Algebra is that there are 10 such values of z – two real ones and four complex conjugate pairs.