MATH 392 -- Seminar in Computational Commutative Algebra
Computing Ideal Intersections
March 29, 2019

Let's compute the intersection of the ideals
$I = \langle x^2 \cdot y - z, x^2 + y^2 + z^2 - 1 \rangle$ and $J = \langle y - x^2, z - x^3 \rangle$
We set up the ideal $t \cdot I + (1 - t) \cdot J$ and eliminate the variable t by computing a lexicographic Gr"obner basis:

$with(Groebner):$
$BIntersection := Basis([t\cdot(x^2\cdot y - z), t\cdot(x^2+y^2+z^2-1), (1-t)\cdot(y-x^2), (1-t)\cdot(z - x^3)], plex(t, x, y, z));$

$$BIntersection := [y^6 + y^4 z^2 - y^3 z^2 - y z^4 - y^4 + y^3 z + y z^2 - z^3, x y^3 z + x y z^3 - y^5 - y^3 z^2 - x y z + x z^2 + y^3 - y^2 z, x y^4 + x y^2 z^2 - y^3 z - y z^3 - x y^2 + x y z + y z - z^2, -y^5 z^3 - y^3 z^5 + x^2 z^5 + 2 y^5 z^2 + 2 y^3 z^4 + y^2 z^5 + z^7 - 2 x^2 z^4 + y^4 z^2 - 2 y^2 z^4 - y z^5 - 2 z^6 + x^2 z^3 - 3 y^5 - 3 y^4 z - 4 y^3 z^2 + y z^4 + 3 x^2 z^2 + x y^3 + x y z^2 + y^4 + 3 y^2 z^2 + 2 y z^3 + 4 z^4 - x^2 z + 3 y^3 - y^2 z - 4 y z^2 - z^3 - x y + x z - y^2 + y z - 3 z^2 + z, y^5 z^2 + y^3 z^4 - x^2 z^4 - y^5 z - y^3 z^3 - y^2 z^4 - z^6 + x^2 z^3 - y^5 - y^4 z - y^3 z^2 + y^2 z^3 + y z^4 + z^5 + x^2 y z + x y^3 + x y z^2 + 2 y^4 + y^3 z + y^2 z^2 + z^4 + x^2 y - 2 x^2 z + y^3 - 2 y^2 z - 2 y z^2 - 2 z^3 - x y + x z - 2 y^2 + 2 y z - z^2 + z, x^2 y^3 + y^5 + y^3 z^2 - x^2 z^2 - y^2 z^2 - z^4 - y^3 + z^2, x^3 z - x^2 y^2 + x y^2 z + x z^3 - y^4 - y^2 z^2 - x z + y^2, y x^3 + x y^3 + x y z^2 - x^2 z - y^2 z - z^3 - x y + z, x^4 + x^2 y^2 + x^2 z^2 - x^2 y - y^3 - y z^2 - x^2 + y, 8 y^5 z^2 + 8 y^3 z^4 - 8 x^2 z^4 - 13 y^5 z - 13 y^3 z^3 - 8 y^2 z^4 - 8 z^6 + 13 x^2 z^3 - 5 y^5 - 8 y^4 z - 5 y^3 z^2 + 13 y^2 z^3 + 8 y z^4 + 13 z^5 + x^2 y^2 - 3 x^2 z^2 + 22 x y^3 + 22 x y z^2 + 21 y^4 + 8 y^3 z + 5 y^2 z^2 - 5 y z^3 + 5 z^4 + x^2 y - 25 x^2 z + 3 y^3 - 22 y^2 z - 18 y z^2 - 30 z^3 - 2 x^2 - 22 x y + 22 x z - 23 y^2 + 25 y z - 4 z^2 + 2 t + 2 y + 22 z] \quad \textbf{(1)}$$

$seq(has(BIntersection[i], t), i = 1..nops(BIntersection));$

$$false, false, false, false, false, false, false, false, false, true \quad \textbf{(2)}$$

This shows that only the last element of the Gr"obner basis contains t. The first 9 polynomials
give a basis for $I \cap J$.

for i **to** $nops(BIntersection) - 1$ **do** $BIntersection[i]$ **end do**;

$$y^6 + y^4 z^2 - y^3 z^2 - y z^4 - y^4 + y^3 z + y z^2 - z^3$$

$$x y^3 z + x y z^3 - y^5 - y^3 z^2 - x y z + x z^2 + y^3 - y^2 z$$

$$x y^4 + x y^2 z^2 - y^3 z - y z^3 - x y^2 + x y z + y z - z^2$$

$$-y^5 z^3 - y^3 z^5 + x^2 z^5 + 2 y^5 z^2 + 2 y^3 z^4 + y^2 z^5 + z^7 - 2 x^2 z^4 + y^4 z^2 - 2 y^2 z^4 - y z^5$$

$$
\begin{gathered}
-2z^6+x^2z^3-3y^5-3y^4z-4y^3z^2+yz^4+3x^2z^2+xy^3+xyz^2+y^4 \\
+3y^2z^2+2yz^3+4z^4-x^2z+3y^3-y^2z-4yz^2-z^3-xy+xz-y^2+yz \\
-3z^2+z \\
y^5z^2+y^3z^4-x^2z^4-y^5z-y^3z^3-y^2z^4-z^6+x^2z^3-y^5-y^4z-y^3z^2+y^2z^3+yz^4 \\
+z^5+x^2yz+xy^3+xyz^2+2y^4+y^3z+y^2z^2+z^4+x^2y-2x^2z+y^3-2y^2z \\
-2yz^2-2z^3-xy+xz-2y^2+2yz-z^2+z \\
x^2y^3+y^5+y^3z^2-x^2z^2-y^2z^2-z^4-y^3+z^2 \\
x^3z-x^2y^2+xy^2z+xz^3-y^4-y^2z^2-xz+y^2 \\
yx^3+xy^3+xyz^2-x^2z-y^2z-z^3-xy+z \\
x^4+x^2y^2+x^2z^2-x^2y-y^3-yz^2-x^2+y
\end{gathered} \tag{3}
$$

$factor(BIntersection[1]);$

$$(y^3-z^2)\,(y^3+yz^2-y+z) \tag{4}$$

What does this tell us??