MATH 392 – Seminar in Computational Commutative Algebra
Fourth Computer Laboratory Day and Problem Set 9
April 8, 2019

Background and Goals

Today's lab, which incorporates Problem Set 9 due on Friday, April 12, will introduce you, via some examples, to:

- Bruno Buchberger's original motivation for the developing the theory of Gröbner bases, and
- Some very interesting connections between *linear algebra* and the algebra of polynomial rings, solving systems of polynomial equations, and related questions.

Lab Questions (With Further Background As Needed)

A. Consider the ideal

$$J = \langle x^2 + y + z^2 - 4, x - y^2 + z^2 + 3, x^2 - yz \rangle.$$

(1) Compute two reduced Gröbner bases for J:
  (a) First, with respect to the `plex(x,y,z)` order in Maple. Call this one *Blex*.
  (b) Second, with respect to the `tdeg(x,y,z)` order in Maple (this is graded reverse lex with $x > y > z$). Call this one *Bgrevlex*.
(2) If you take any polynomial $f \in \mathbf{Q}[x, y, z]$ and compute its remainder on division by *Blex*, using the `plex(x,y,z)` monomial order, what monomials will appear in the remainder? (You can see this by listing out the leading terms of each of the basis polynomials, then thinking: what monomials are *not divisible* by any of those leading terms.) How many such monomials are there?
(3) Same questions as in (2) for the remainder on division of f by *Bgrevlex* using the `tdeg(x,y,z)` order.

Your answers for the *numbers* of monomials in parts (2) and (3) of question A should be *the same*, but the monomials themselves should be different depending on which monomial order you are using. This is no accident because both collections of monomials give bases for vector spaces over $\mathbf{Q}$ that are *isomorphic* because both are different representations of *the same algebraic structure*, a new example of a ring called the *quotient ring* $\mathbf{Q}[x, y, z]/J$ (careful: even though the same word is used, this is *not* directly related to the quotient or colon ideals that we discussed last week!) Buchberger's original motivation for developing Gröbner bases was to give a concrete way to do computations in these quotient rings. To be more precise about what these structures are:

- Pick either of the two Gröbner bases above.
- The elements of the quotient ring can be thought of as all the linear combinations of the monomials that are *not in* $\langle LT_>(J) \rangle$: that is, all possible remainders on division by the Gröbner basis with respect to that monomial order $>$.

- The sum operation in the quotient ring is just the ordinary sum of the linear combinations of monomials or remainders as in the previous bullet.
- The product operation is defined like this: To multiply two linear combinations of the set of monomials, by the distributive law, it suffices to know how to multiply all pairs of the basis monomials. (See Chapter 5 in the text for more details about how this works.) If the product of two of the basis monomials is another of the basis monomials, then that is the product in the quotient ring. On the other hand, if the product is not one of the basis monomials (if it is divisible by one of the leading terms of the elements of G), then to find the product in the quotient ring, you compute the *remainder of the product with respect to the Gröbner basis.* In either case you can phrase the definition of the product operation like this: in $Q[x, y, z]/J$,

  $$x^a y^b z^c \cdot x^d y^e z^f = \overline{x^{a+d} y^{b+e} z^{c+f}}^G$$

  where G is the appropriate Gröbner basis, depending on which monomial order and which set of monomials you are looking at.

B. Compute the "multiplication tables" for both sets of basis monomials from question A using the product operation defined above. Put nicely formatted versions in your solution, either by entering everything into a text region in your Maple worksheet, or by writing out the tables by hand on paper.

One of the amazing things about these quotient rings is how they encode information about the variety $V(J)$ in linear algebraic form! Here is some information about computations with linear algebra in Maple. The commands you will need are all in the `LinearAlgebra` package. Load this using the `with` command as we have seen for other packages. You define matrices in Maple with commands like this:

```
A := Matrix([[1,2],[3,4]]);
```

The output will show you exactly what is happening here and how you will enter other square matrices. If A is a square matrix, then

```
CharacteristicPolynomial(A,t);
```

computes the *characteristic polynomial* $\det(A - tI)$ as a polynomial in the variable t. The roots of the characteristic polynomial are the *eigenvalues* of the matrix A. You can find them by either using `fsolve` on the characteristic polynomial, or by using

```
evalf(Eigenvalues(A));
```

(Note: The `evalf` converts the output of `Eigenvalues` to decimal form. The results here can be slightly different because the *fsolve* command and the `Eigenvalues` command are using different sorts of numerical approximation algorithms.)

C. Use the lex Gröbner basis $Blex$ to find all of the points of $V(J)$, real and complex.

D. List the basis monomials for the quotient ring in any one order and *use that consistently*. Now, using the grevlex Gröbner basis $Bgrevlex$, multiply each of the basis monomials for the quotient ring by x and take the remainder on division by $Bgrevlex$. Pick off the coefficients of the remainders and put them into the columns of a square matrix in the order corresponding to your ordering on the basis monomials (one column for the product of x and each basis monomial). As you will probably recognize from Linear Algebra, the matrix you are computing here is the *matrix* $[M_x]$ *of a linear mapping* M_x on $\mathbf{Q}[x,y,z]/J$ defined by

$$\begin{aligned} M_x : \mathbf{Q}[x,y,z]/J &\to \mathbf{Q}[x,y,z]/J \\ f &\mapsto x \cdot f, \end{aligned}$$

using the product operation in the quotient ring described above, and with respect to the (ordered) monomial basis for the quotient ring (in the domain and the target space of the linear mapping). Now do the same for the products by y and z and derive the matrices of the corresponding M_y and M_z multiplication mappings.

E. Find the eigenvalues of each of the matrices of $[M_x], [M_y], [M_z]$. Compare with your results from question C. What can you say here?

What you are seeing in this example is how (one form of) a general result known as *Stickelberger's Theorem* applies to this ideal J. If you want to see a proof why this works, consult Chapter 2 of the book *Using Algebraic Geometry* by the same team of authors who brought you *Ideals, Varieties, and Algorithms*!