MATH 392 – Seminar in Computational Commutative Algebra
First Computer Laboratory Exercises
February 4, 2019

Background and Goals

A large part of our work this summer will be carrying out calculations of Gröbner bases and other computations within the computer algebra system called *Maple*. This week, we will want to get familiar doing mathematics in this context.

Note: Several other mathematical software packages, including the freeware Sage and Macaulay 2 can also be used for these, although the syntax of commands is different and Macaulay 2 has no builtin graphics. If you want to get copies of either of these for your own use, let me know and I can help you with installation, how those programs work, etc.

General Information about Maple

Launching Maple

To get into Maple, you will need to:

1) Log onto the campus network with your username and password.
2) You should now see the lab desktop with "icons" (technically, "shortcuts" in Windows) for Maple, etc. You will double click on the Maple icon.
3) From the Maple opening screen, select *New Worksheet* to produce a file that behaves as described below. (The *New Document* produces a similar workspace, but without the input prompts, and set up rather differently.)
4) After a few more seconds you should see a new Maple window, with a "subwindow" marked Untitled (1) opened inside it. The Untitled (1) window is a blank *Maple worksheet*.

Maple Worksheets

Worksheets are integrated text/graphics/mathematics documents where any or all of the following can be done:

a) you can type in commands from the keyboard to ask Maple to perform many different kinds of calculations, read data from external files, store work in files, etc.
b) output generated by Maple from your input commands (numerical values, symbolic formulas, and graphics) will be displayed,
c) you can modify commands, generating new output, store your results for use later, etc.
d) you can enter text to annotate and explain the results of computations.

Take a few seconds and notice the features of this window – especially the File, Edit, View, Insert, etc. pull-down menus across the top, the "tool bar" below them with the icons for various operations (can you guess what each of them does?) the "scroll bar" on the right that you can use to move around within the worksheet, to see previous input and output lines, etc. Most of the common operations appear both as items in one of the pull-down menus and as toolbar buttons. The toolbar buttons are faster to use, but somewhat limited in the options they offer, so you will want to have a general idea what is accessible from where in the pull-down menus. Take a few minutes and look these over.

Input and Text Regions in a Worksheet

Now look at the Untitled (1) window itself. On the first line of the worksheet, you should see a [> in red. This is the Maple *input prompt* – the signal that Maple is ready to accept a command from you and try to execute it. A new input prompt will be generated automatically in a worksheet each time you enter a command and execute it *at the end of a worksheet in progress*. You can also go back and insert new commands and output at any point in a worksheet as follows. You can *insert an input prompt* and an input region at any point in the worksheet by placing the cursor at the desired location and pressing the toolbar button marked with [>. (After executing an inserted command like this, Maple will drop to the next input line, so to insert several input lines in the middle of a worksheet, you will need repeat the above.)

As we mentioned before, Maple worksheets can contain text as well as commands and output. To create a text region you can press the [> button above, then the button marked by a capital T (this changes the region into a text region). Any text can now be entered, and this will be treated as an "inert" comment. That is, it will appear as you enter it when you print out the worksheet, and it will not be treated as Maple input.

Saving and Reloading your Maple Worksheets

When you begin working on a worksheet, you will usually want to *save your work* every once in a while in case a computer problem develops, or in case you need more than one lab session to complete the work you are doing. This can be done most directly by saving to your network drive following these directions:

2) Select the SAVE option from the FILE pull-down menu, or click the toolbar button between the open folder and the printer icons. (The button shows one of the ancient 3 1/2 "floppy" removable storage diskettes – none of you probably ever used one of those since they became obsolete around 2000!)
3) If you are saving your work for the first time, you will see the SAVE AS dialog box. Highlight your network drive, then go to the File Name box, and type in a name for the file where the worksheet will be stored (it's safest to avoid using spaces in filenames!). Then click the OK button to write the worksheet to the network drive. (You will only need to type the filename in once in a session. Subsequent saves just update the file.)
4) When you have the worksheet saved as you want it, you can exit Maple.

5) To update the worksheet further in a later session, you can read the worksheet back into Maple using the OPEN option from the FILE Menu, or the "open folder" toolbar button. Get into Maple as above, and select OPEN from the FILE Menu. Maple will prompt you as above for the name of the worksheet with a dialog box very like the SAVE AS box described above. Highlight the appropriate network drive, then the worksheet you want, and click OK. IMPORTANT NOTE FOR FUTURE REFERENCE: When you read a previously created worksheet into a new Maple session, you *have not* actually executed any of the commands in it in the new session. If you want to make use of any results in that worksheet, you will need to execute the commands again. In the EDIT pull-down menu, you will see an EXECUTE option that lets you recompute an entire worksheet or a section of one.

Printing

OPEN the worksheet you want to print, and press the toolbar icon that looks like a printer, and press OK on the PRINT dialog box (the settings should be set up correctly to print automatically). Your output should appear shortly on the printer.

Getting Out

When you leave, quit the Maple window (FILE/EXIT) or the X button on the top of the window, and log off the campus network.

A First Sample Maple Session

Let's get right down to work and walk through a first sample sample session! First, you will need to get into Windows and Maple as described in the General Information section above.

2D Plotting

The basic Maple command for 2D plotting, used for graphs of the form $y = f(x)$ and for parametric curves $\alpha(t) = (x(t), y(t))$ is called `plot`. The basic format for graphs $y = f(x)$ is

```
plot(function,range,options);
```

You fill in the information as follows to generate the plot you want:

1) `function` is the function to be plotted – the simplest way to specify one is via a formula for $f(x)$ (an *expression* in Maple)
2) `range` is the range of x-values you want to see plotted, entered in the form `x = low..high`, and
3) `options` can be used to control the form of the plot if desired. No options need be specified however if you don't want to. More on this later.

The basic format for parametric curves $\alpha(t) = (x(t), y(t))$ is

```
plot([x-comp,y-comp,t-range],options);
```

where

1) `x-comp` and `y-comp` are the x- and y-component functions of the curve to be plotted (each of these can be an *expression* involving the variable (parameter) t,
2) `t-range` is the range of t-values you want to see plotted, and
3) `options` can be used to control the form of the plot if desired. No options need be specified however if you don't want to.

Note that the x- and y-components and the t-range all go inside a set of square brackets $[,]$, separated by commas. This is also Maple's notation for a *vector, or ordered list*.

Example 1

Enter commands like this to plot functions of x:

```
plot(3*sin(x) - 4*cos(x),x = -2*Pi..2*Pi);
```

Next, let's see the parametric curve $\alpha(t) = (t^2 - 1, t^3 - t)$ from class. Using the `plot` command, as described above, display a plot of this curve for $t \in [-3, 3]$, using

```
plot([t^2 - 1,t^3 - t,t = -3..3]);
```

What is a Maple Command?

It may help to think of one as a *program* or *function of one or more variables* that Maple knows how to compute, given "inputs" from you, although you must then keep straight the distinction between these functions Maple knows how to execute, and the functions or formulas you enter for Maple to work with!

Now that we have seen some first examples of Maple commands, here is some more information about the *syntax rules* that Maple uses to decide if what you have typed in is a well-formed command it can execute, and what the *semantics* or meaning of the commands are.

Almost all of the built-in Maple commands have a syntax like the `plot` command discussed above. The format is either

`name(values);`   or   `name(values):`

In this general description, `name` is the name of the command or function (like "plot" yesterday), there is a matching set of open and close parentheses following the name, and they surround the `values` – the input from you the command needs to do its job (like the formula of the function to be plotted and the range of x-values you want). The values are listed separated by commas within the parentheses. The two forms of the command above tell Maple to execute the command when you press ENTER on that input line. In the first case – ending the command with a semicolon (;) – you are instructing Maple to display the output. With colon (:), Maple will execute the command but not display the output (this is useful sometimes for intermediate steps in a big computation where you don't want or don't need to see the output).

One frequent source of errors in Maple sessions is unmatched parentheses, (or square or curly brackets, etc.) One "hacker's trick" is to count the open parens, count the close parens and make sure you get the same number (of course, if you don't, you also have to decide where to insert the missing one ...).

Getting Help On-Line

The exact format Maple is expecting for each type of command is specified in the programming of the Maple system. Usually, there is *little or no freedom* in the ordering of what has to go where and in what format in the list of values. Much useful information on this for all the built-in commands, and LOTS of instructive examples are contained in the Maple on-line HELP facility. For example try looking at the on-line help page for the `plot command`. (From the HELP pull-down menu, select Topic Search, type in the word "plot," and press OK.) Note the example plot commands at the bottom and the links to related help pages.

Maple Expressions

Formulas or expressions are entered in something like usual mathematical notation:

1) The symbols for addition, subtraction, multiplication, and division are $+, -, *, /$ respectively.
2) The caret (ˆ) is the Maple symbol for raising to a power.
3) The asterisk symbol for multiplication MUST be included whenever you are performing a product in a formula. (It shows up as the "dot for multiplication" on the screen.) Moreover, everything must be entered in one linear string of characters, so you will need to use *parentheses* to group terms to get the expressions you want. The rule to keep in mind is: *Maple always evaluates expressions by doing powers first, then products and quotients, then sums and products, left to right, unless parentheses are used to override these built-in rules*. For example, the Maple expression `a + b^2/c + d` is the same as the mathematical formula:

$$a + \frac{b^2}{c} + d.$$

If you really wanted

$$\frac{a + b^2}{c + d}$$

you will need to enter the expression `(a + b^2)/(c + d)`. What if you really wanted:

$$\frac{(a + b)^2}{c + d}?$$

4) Maple "knows" all the usual elementary functions from calculus. The names of the most common ones are `sin, cos, tan, exp, ln, sqrt`. To use one of these functions in a Maple formula, you put the name, followed by the "argument" (that is, the constant or expression you are applying the function to) *in parentheses*.

Example 2

How would you enter each of the following formulas as Maple expressions? Try typing each in as a Maple input line, and compare with the output they generate:

1) $4xy^2 - 2x^3 + 3y - 7$

2) $1 + \frac{1}{x^2} - \frac{x+2}{x^4}$

3) $\frac{-b+\sqrt{b^2-4ac}}{2a}$

Implicit 2D Plotting

The 2D plots we generated yesterday were all either graphs of functions $y = f(x)$, or parametric curves. In addition to these, we will also want to be able to plot the affine variety

$$\mathbf{V}(g(x,y)) = \{(x,y) \in \mathbf{R}^2 : g(x,y) = 0\}$$

for a general polynomial $g(x,y)$ in two variables. The technical name for this in Maple is *implicit curve plotting*, since the equation $g(x,y) = 0$ defines one or more functions $y = f(x)$ "implicitly." To plot these curves in Maple, we need a new command called `implicitplot`. This command is not part of the basic Maple system; instead it is a part of a separate *package* of commands for various plotting operations called the `plots` package. Before you can use this command, you will need to "load" the plots package with the command

```
with(plots);
```

(The output is the list of commands in the package; you should see `implicitplot` about halfway along the list. If you don't want to see this each time, you can also put a colon after the `with` command as described above.)

The format for the `implicitplot` command is

```
implicitplot(expression,x-range,y-range);
```

The expression should be the equation $g(x,y)$ (Maple assumes you mean to plot $g(x,y) = 0$). This command generates a plot of the part of the variety $\mathbf{V}(g(x,y))$ in the rectangular box in the plane defined by the x- and y-ranges.

Example 3

a) For instance, try the following command:

```
implicitplot(x^3 - 3*x*y^2 - 3,x=-3..3,y=-3..3);
```

Try adding a text region answering the following question: How many different "implicit functions" $y = f(x)$ are defined by the equation $x^3 - 3xy^2 - 3 = 0$? Also practice plotting these varieties:

b) $\mathbf{V}(y^2 - x^3 + 9x)$
c) $\mathbf{V}(x^3 - 3x + 2xy - y^4 - 1)$

3D Curve Plotting

The `spacecurve` command in the `plots` package can be used to draw parametric curves in 3D. The format for the command is

```
spacecurve([x(t),y(t),z(t),t=a..b]);
```

where $x = x(t)$,$y = y(t)$,$z = z(t)$ are the parametric equations of the curve and the range of parameter values to be plotted is $a \leq t \leq b$. (Note the placement of the square brackets which follows the same pattern as the parametric version of `plot`!) For instance, try entering

```
spacecurve([t,t^2,t^3,t=-2..2]);
```

to plot the *twisted cubic* curve from class.

The first plot you see here might be rather uninformative. Fortunately, Maple also lets you add axes and look at a 3D plot from different viewpoints. Here's how:

1) Click the left mouse button once over the graph. This should bring up a "bounding box" for the graph with corners you can *drag and drop* with the mouse to resize the graph.
2) If you place the mouse over the graph now and hold the left button down, when you move the cursor, you should see a box rotating following the cursor position. Release the left mouse button when you get to the position you want.
3) If you click the right mouse button, you will bring up a menu containing the options `Redraw, Copy, Style, Axes, Color` and several others. Select `Axes, Normal` with the left mouse button. Then repeat clicking the right mouse button and select `Redraw`.

Practice a few times repositioning the viewpoint (rotating the viewing box) and redrawing the graph.

To draw several parametric curves in the same graph, you can include several curve specifications in the form `[x(t),y(t),z(t),t=a..b]` between curly braces $\{,\}$, separated by commas inside the parentheses for the `spacecurve` command.

Surface Plots

a. Plotting a graph $z = f(x,y)$.

The command for 3D plotting is called `plot3d` (naturally enough!) Its format is similar to, but not exactly the same as the format of `plot`. To draw a graph with `plot3d`, you use a command of the format:

```
plot3d(function,xrange,yrange,options);
```

The `function` is the function $f(x, y)$, entered in the usual Maple syntax for expressions. The `xrange` and `yrange` specify a rectangular box in the plane that the plot will be constructed over; the `options` can be used to specify how the plot is drawn. Look at the online help for `plot3d[options]` if you want to see what things are possible. Most of the options can also be invoked from the 3D graphics output via the pop-up menu, or the toolbar buttons. Using the PATCHCONTOUR style is often a very effective way to visualize the shape of a surface.

For example, try

```
plot3d(-x^4+2*x^2-x/2-y^2-1,x=-2..2,y=-2..2);
```

to plot the surface

$$z = -x^4 + 2x^2 - x/2 - y^2 - 1.$$

One very nice feature of Maple 3D graphics is that you can change viewpoint, add axes, etc. using the mouse and the toolbar options in the 3D graphics window. The method is the same as that for the `spacecurve` described above.

Example 4

We will build up a "composite" Maple graphics display by combining a plot of a graph $z = f(x, y)$ produced with `plot3d` and a plot of a parametric curve produced using `spacecurve`. This can be done via the `display3d` command in the `plots` package. First we generate the plot of $z = x^2 - y^2$ for $(x, y) \in [-3, 3] \times [-3, 3]$, and assign the output to the variable `P1`.

```
P1:=plot3d(x^2-y^2,x=-3..3,y=-3..3):
```

(The colon at the end of the input line *suppresses the output*; if you want to see the output replace the colon with a semicolon.) Then, we generate a plot of the parametric curve $\alpha(t) = (t, t^2, t^3)$ for $t \in [-2, 2]$, drawn with a relatively thick line and in black for contrast, and put the results into the variable `P2`:

```
P2:=spacecurve([t,t^2,t^3,t=-2..2],color=black,thickness=2):
```

Finally, we display both graphs together:

```
display3d({P1,P2});
```

(Here of course, you *do* want to see the output, presumably.) Maple is smart enough about these plots to combine the scales correctly and show the curve and surface in their proper relative positions! Look at this one for a while until you see how the curve and the surface are intersecting. How many intersection points are there?

Exercises

Do the following exercises in a new worksheet (i.e. separate from your work on the examples above), print and submit them with Problem Set 2, due Friday, February 8.

Exercise 1

The line segment from (a, b, c) to (d, e, f) can be parametrized as follows

$$\alpha(t) = (a + t(d - a), b + t(e - b), c + t(f - c))$$

for $0 \leq t \leq 1$. The points $P = (0, 0, 0)$, $Q = (2, 0, 0)$, $R = (0, \sqrt{3}, 1)$, $S = (0, -1, \sqrt{3})$ are four corners of a cube in $\mathbf{R}^3$ with edges PQ, PR, and PS, because the vectors $\vec{u} = Q - P, \vec{v} = R - P, \vec{w} = S - P$ all have magnitude 2, and $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 0$. Your assignment, should you decide to accept it (just kidding!), is to create a picture of this cube by drawing the 12 edges together on the same set of axes in $\mathbf{R}^3$ (begin by drawing the line segments from P to Q, P to R, and P to S). Also explain how you found the other 4 corners of the cube. This can be done in a text region.

Any number of different 3D plots can be displayed together by assigning each to a variable, then putting the set or list of plot variables names in the `display3d` command.

Exercise 2 – Implicit Surface Plots

To plot an surface defined as a variety $\mathbf{V}(g(x, y, z))$. you can use the `implicitplot3d` command. Look up the on-line help for this command and use it in the following exercise. Generate a plot of the following variety:

$$S = \mathbf{V}(z^2 - (16 - x^2 - y^2)((x + 2)^2 + y^2 - 1)((x - 2)^2 + y^2 - 1)/10)$$

and explain the shape you see. You will want to "walk around" this one a lot by rotating and looking at it from different viewpoints (For example, which (x, y) correspond to points on S and why? You will also need to think about how you choose the x-, y-, and z-ranges.) Suggestion: Use the options `style=PATCHCONTOUR` and `grid=[30,30,30]` to see detail. Be aware though that this larger than normal numbers of grid points will mean the surface takes longer to compute and "render" or draw.