Mathematics 392–Seminar in Computational Commutative Algebra
Course Information–Fall 2006

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Office Hours: MWF 10-11, TR 1-3, and by appointment.

Course Description and Topics to be Covered

The Seminar in Computational Commutative Algebra is a one-semester advanced special topics course. Commutative algebra is the study of a class of algebraic structures called *commutative rings* (and other structures that are built up from them). A ring is a set R with two operations $+, \cdot$ that satisfy all of the usual rules of algebra, except that multiplication need not be commutative, and not all nonzero elements necessarily have multiplicative inverses. A typical example is the ring of integers, $\mathbf{Z}$, that you studied in MATH 243 (Algebraic Structures). A *commutative ring* is a ring in which the multiplication operation is commutative. So for instance, the $n \times n$ matrices with real entries, $M_{n\times n}(\mathbf{R})$ is a ring, but not a commutative ring. $\mathbf{Z}$ is a commutative ring. Other especially important examples of commutative rings are polynomial rings in one and several variables (and also some closely related rings of rational functions and formal power series).

One reason polynomial rings and their relatives are particularly interesting and worthy of intensive study is that there are many links between the algebra of these rings and the *geometry* of *algebraic varieties*–sets defined as the solutions of systems of polynomial equations.

In this course we will pursue this relation is some depth, together with some recently-developed algorithmic techniques (Gröbner bases) implemented in computer algebra systems such as Maple and Magma. To give the idea in a nutshell, an *ideal* I in a ring R is a subset that is closed under sums and also closed under multiplication by all elements of R. A Gröbner basis for an ideal is a special generating set for an ideal that generalizes the row-reduced echelon form for a system of linear polynomials, and also the greatest common divisor of a collection of polynomials in one variable. Each ideal I has many different Gröbner bases, since the definition involves the choice of a *monomial order* – a way of ordering the terms in polynomials that is consistent with the multiplication of polynomials.

If we know a Gröbner basis for a given ideal, then we have much information about it. Gröbner bases with respect to some monomial orders include polynomials where some variables have been eliminated, for instance. Any Gröbner basis lets us test arbitrary polynomials for membership in the ideal via a general polynomial division process, and so forth.

One of our main goals in the first section of the course will be to develop *Buchberger's algorithm* – a procedure that computes a Gröbner basis for an ideal starting from any generating set for the ideal.

We will also look at a few of the ways this algorithm, and others based on it, have been used in applications to geometry, robotics, and other areas.

The topics we will be studying are

- Unit I: The basic language of varieties, ideals, and algorithms (about 6 class days)
- Unit II: Gröbner bases (about 8 days)
- Unit III: Elimination theory, resultants, and geometric applications (about 7 class days)
- Unit IV: "The algebra-geometry dictionary" (about 7 class days)
- Unit V: Applications to quotient rings (about 3 class days)
- Unit VI: Applications to robotics and geometric theorem-proving (about 6 days)

Text

The text for the course is *Ideals, Varieties, and Algorithms*, 2nd edition[1] by Cox, Little, and O'Shea. (Catchy title!) We will cover much of the material in Chapters 1-6 this semester. Please feel free at any time to direct any comments, complaints, etc. about the book to me.

Assignments and Grading

The only way to really learn advanced mathematics is to work out and present solutions to challenging problems. Thus the focus of this course will be a series of problem sets, given out roughly weekly. Some of these will begin with a computer lab session in the Swords 219 Unix (now Linux!) lab, and continue to an individual problem set.

In addition, there will be two larger "summary" problem sets, one at midterm, the other at the end of the semester. There will be no in-class exams, but over the course of the semester I will ask each student to give two (short–approximately 15 minute) oral

[1] Important Note: Check your copy of the text carefully. It should say "2nd edition, sixth printing." If it says "Corrected fifth printing," please bring it to my attention immediately, and I will arrange to replace it. The "corrected" fifth printing contained a large number of insidious and almost inexplicable errors introduced by the publisher's book-printing software. It should have been entirely destroyed, but a few copies "escaped" even so(!)

presentations to the seminar, either on problem solutions or perhaps on the proof of a particular result from the text. The problems will come from the problem sets, and I will assign the presentations when the problem set goes out. You will always have adequate lead time to prepare for these presentations and consult with me if necessary.

Beside the problem sets, the other assignment for the course will be a *final project.* You will work on these projects in pairs, give a presentation to the seminar on the project, and hand in a written summary of your presentation to me. The topics for these presentations will be extensions or conclusions of subjects we study in class. I will distribute more information about these projects later in the semester.

Your final grade for the seminar will be computed using the following weight factors:

1.	*Weekly problem sets*	30% of final grade
2.	*Midterm problem set*	15%
3.	*Final problem set*	25%
4.	*In-class oral presentations*	10%
5.	*Final project/presentation*	20%

If you ever have a question about an assignment, or about your standing in the course, please feel free to consult me, either during regular office hours, or by making an appointment to see me.