Mathematics 392 – Seminar in Computational Commutative Algebra
Comment on Additional Problem, Problem Set 9
November 6, 2006

There was quite a bit of confusion about how the earlier parts of the Additional Problem on this assignment would give a way to compute resultants. Here is the idea for part D and an example that is somewhat more representative than part E.

D) Suppose we perform the following divisions in the course of computing $\gcd(f, g)$ by the Euclidean algorithm (starting with $\deg(g) > \deg(f)$):

$$\begin{aligned} g &= q_1 f + r_1 \\ f &= q_2 r_1 + r_2 \\ &\vdots \\ r_{k-2} &= q_k r_{k-1} + r_k \end{aligned}$$

where r_k is the last nonzero remainder. Say the leading coefficients of f, r_1, r_2, etc. are $a_0, a_0^{(1)}, a_0^{(2)}$, respectively. Then applying parts B (division) and C (swap to put the polynomial of smaller degree first) repeatedly we have

$$\begin{aligned} \operatorname{Res}(f, g, x) &= a_0^{\deg(g)-\deg(r_1)} \operatorname{Res}(f, r_1, x) \\ &= a_0^{\deg(g)-\deg(r_1)} (-1)^{\deg(f)\deg(r_1)} \operatorname{Res}(r_1, f, x) \\ &= a_0^{\deg(g)-\deg(r_1)} (-1)^{\deg(f)\deg(r_1)} \left(a_0^{(1)}\right)^{\deg(f)-\deg(r_2)} \operatorname{Res}(r_1, r_2, x) \\ &= a_0^{\deg(g)-\deg(r_1)} (-1)^{\deg(f)\deg(r_1)} \left(a_0^{(1)}\right)^{\deg(f)-\deg(r_2)} (-1)^{\deg(r_1)\deg(r_2)} \operatorname{Res}(r_2, r_1, x) \\ &= \cdots \end{aligned}$$

The process terminates either when the remainder r_k is a non-zero constant (and then we use part A), or else if we get r_k of positive degree and the next remainder is zero (in which case $\operatorname{Res}(f, g, x) = 0$ since f, g have a common factor of positive degree – the polynomial $\gcd(f, g)$). Here is an example showing how this works in practice.

Let $f = x^2 + 2x + 3$ and $g = x^3 + 7x^2 + x + 2$. Then the steps of the Euclidean algorithm look like

$$\begin{aligned} g &= (x+5)f + (-12x - 13) &&\Rightarrow r_1 = -12x - 13 \\ f &= (-1/12x - 11/144) r_1 + 289/144 &&\Rightarrow r_2 = 298/144 \\ r_1 &= q_3 r_2 + 0 \end{aligned}$$

Then

$$\begin{aligned} \operatorname{Res}(f, g, x) &= (1)^{(3-1)} \operatorname{Res}(f, r_1, x) \quad \text{(part B)} \\ &= (1)^{(3-1)} (-1)^{2\cdot 1} \operatorname{Res}(r_1, f, x) \quad \text{(part C)} \\ &= (1)^{(3-1)} (-1)^{2\cdot 1} (-12)^{(2-0)} \operatorname{Res}(r_1, r_2, x) \quad \text{(part B)} \\ &= (1)^{(3-1)} (-1)^{2\cdot 1} (-12)^{(2-0)} \cdot (289/144)^{(1-0)} \quad \text{(part A)} \\ &= 289 \end{aligned}$$

It is easy to check that this is exactly the determinant of the Sylvester matrix for f, g:

$$\det \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 7 & 1 \\ 3 & 2 & 1 & 1 & 7 \\ 0 & 3 & 2 & 2 & 1 \\ 0 & 0 & 3 & 0 & 2 \end{pmatrix} = 289$$